

EE391D Advanced Topics in Photonics

Light propagation in anisotropic crystals lesson 1

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Outline

- 1 Energy and symmetry considerations
- 2 Classification of anisotropic crystals
- 3 Plane waves in anisotropic crystals
- 4 Reference texts

Energy and symmetry considerations

In a generic anisotropic medium, the relationship between the polarization vector \mathbf{p} and the electric field \mathbf{e} reads as $\mathbf{p} = \epsilon_0 \underline{\underline{\chi}} \mathbf{e}$, with:

$$\underline{\underline{\chi}} = \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix}, \quad (1)$$

being $\underline{\underline{\chi}}$ a rank-2 tensor describing the anisotropic response of the material. The electric displacement then becomes:

$$\mathbf{d} = \epsilon_0 \mathbf{e} + \mathbf{p} = \underline{\underline{\epsilon}} \mathbf{e} \quad (2)$$

with:

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}, \quad (3)$$

the anisotropic dielectric tensor.

Energy and symmetry considerations

- Question: are all the 9 elements of the tensors $\underline{\underline{\chi}}$ and $\underline{\underline{\epsilon}}$ independent?

Energy and symmetry considerations

- Question: are all the 9 elements of the tensors $\underline{\chi}$ and $\underline{\epsilon}$ independent?

To answer to this question we can start from the electromagnetic energy stored in the material:

$$\mathcal{E} = \frac{1}{2} (\mathbf{e} \cdot \mathbf{d} + \mathbf{h} \cdot \mathbf{b}) = \frac{1}{2} (e_k \epsilon_{kl} e_l + \mu_0 h_l h_l), \quad (4)$$

where we have adopted Einstein summation convention over repeated indices, i.e., $\sum_{j=1}^3 a_{ij} b_j \equiv a_{ij} b_j$. We then take the time derivative of (4):

$$\dot{\mathcal{E}} = \frac{\epsilon_{kl}}{2} (\dot{e}_k e_l + e_k \dot{e}_l) + \mu_0 \dot{h}_l h_l, \quad (5)$$

where $\dot{A} \equiv \frac{\partial A}{\partial t}$. From the conservation of energy of Maxwell equations:

$$\dot{\mathcal{E}} + \nabla \cdot (\mathbf{e} \times \mathbf{h}) = 0, \quad (6)$$

Energy and symmetry considerations

we have:

$$\frac{\epsilon_{kl}}{2} (\dot{e}_k e_l + e_k \dot{e}_l) + \mu_0 \dot{h}_l h_l = -\nabla \cdot (\mathbf{e} \times \mathbf{h}). \quad (7)$$

From the Poynting theorem of Maxwell equations:

$$\nabla \cdot (\mathbf{e} \times \mathbf{h}) + \mathbf{e} \dot{\mathbf{d}} + \mathbf{h} \dot{\mathbf{b}} = \mathbf{0}, \quad (8)$$

by substituting into (7), we obtain:

$$\frac{\epsilon_{kl}}{2} (\dot{e}_k e_l + e_k \dot{e}_l) + \mu_0 \dot{h}_l h_l = \epsilon_{kl} \dot{e}_l e_k + \mu_0 \dot{h}_l h_l, \quad (9)$$

which implies:

Symmetry condition

$$\epsilon_{kl} = \epsilon_{lk}$$

The dielectric tensor has only 6 independent elements.

Energy and symmetry considerations

- Question: can we simplify more the expression of $\underline{\underline{\epsilon}}$ for our study?

Energy and symmetry considerations

- Question: can we simplify more the expression of $\underline{\underline{\epsilon}}$ for our study?

We can start from the expression of the electric energy density

$$\mathcal{W}_e = \frac{1}{2} \epsilon_{kl} e_l e_k = \frac{1}{2} \mathbf{e}^T \cdot \underline{\underline{\epsilon}} \cdot \mathbf{e}, \quad (10)$$

which is a quadratic form. In (10), T is the transpose operator and \cdot indicates the matrix product. Due to the Hermitian nature of the dielectric tensor $\underline{\underline{\epsilon}} = \underline{\underline{\epsilon}}^\dagger$, we can diagonalize $\underline{\underline{\epsilon}}$ by a similarity transformation

$\underline{\underline{\gamma}} = \underline{\underline{t}}^T \underline{\underline{\epsilon}} \underline{\underline{t}}$, with $\underline{\underline{t}}$ a new reference system:

$$\mathbf{e} = \underline{\underline{t}} \mathbf{e}', \quad (11)$$

we have:

$$\mathcal{W}_e = \frac{1}{2} \mathbf{e}'^T \cdot \underline{\underline{t}}^T \underline{\underline{\epsilon}} \underline{\underline{t}} \cdot \mathbf{e}' = \frac{1}{2} \mathbf{e}'^T \cdot \underline{\underline{\gamma}} \cdot \mathbf{e}', \quad (12)$$

Energy and symmetry considerations

with diagonal $\underline{\underline{\gamma}}$:

$$\underline{\underline{\gamma}} = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix}. \quad (13)$$

From (12), $\underline{\underline{\gamma}}$ represents the expression of the new dielectric tensor in the reference system described by $\underline{\underline{t}}$, which is called **principal axis reference**. In order to construct $\underline{\underline{t}}$ and $\underline{\underline{\gamma}}$, we begin by calculating the eigenvalues and eigenvectors of $\underline{\underline{\epsilon}}$:

$$\underline{\underline{\epsilon}}\mathbf{c}_j = \alpha_j\mathbf{c}_j, \quad (14)$$

with α_j the eigenvalue and \mathbf{c}_j the eigenvector of $\underline{\underline{\epsilon}}$, with $j = 1, 2, 3$. Since the matrix $\underline{\underline{\epsilon}}$ is hermitian, eigenvalues are reals and eigenvectors are orthogonal to each other. The transformation matrix $\underline{\underline{t}}$ is then constructed from:

$$\underline{\underline{t}} = [\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3] \quad (15)$$

Energy and symmetry considerations

- Exercise: Calculate the expression of $\underline{\underline{\gamma}}$

Answer: if the eigenvectors are normalized, i.e., if $\mathbf{c}_i \mathbf{c}_j = \delta_{ij}$, we can verify that:

$$\underline{\underline{\gamma}} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}. \quad (16)$$

- Advanced Question: What are the physical implications of the similarity transformation $\underline{\underline{\gamma}}$ on the electromagnetic fields \mathbf{e} and \mathbf{h} and why is it called similarity transform?

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Classification of anisotropic crystals

The analysis described above shows that only a maximum of 3 elements of the dielectric tensor can be really independent. Only the following cases are then possible for an anisotropic crystal:

- $\alpha_1 = \alpha_2 = \alpha_3 \rightarrow$ This represents an isotropic material.
- $\alpha_1 = \alpha_2 \neq \alpha_3 \rightarrow$ This indicates a **Uniaxial crystal**. The direction corresponding to α_3 known as **optical axis**.
- $\alpha_1 \neq \alpha_2 \neq \alpha_3 \rightarrow$ This is a **Biaxial crystal**.

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Plane waves in anisotropic crystals

Starting from Maxwell equations:

$$\begin{cases} \nabla \times \mathbf{h} = \partial_t \mathbf{d} \\ \nabla \times \mathbf{e} = -\partial_t \mathbf{b} \end{cases}, \quad \partial_t \equiv \frac{\partial}{\partial t}, \quad (17)$$

and looking for plane waves solutions $\mathbf{e} = \mathbf{E}e^{i\omega t - i\mathbf{k}\mathbf{r}}$, $\mathbf{h} = \mathbf{H}e^{i\omega t - i\mathbf{k}\mathbf{r}}$, we obtain:

$$\begin{cases} \mathbf{k} \times \mathbf{H} = -\omega \mathbf{D} \\ \mathbf{k} \times \mathbf{E} = \omega \mu_0 \mathbf{H} \end{cases}, \quad (18)$$

From these equations we observe that the field \mathbf{D} is orthogonal to \mathbf{k} and \mathbf{H} , while the field \mathbf{H} is orthogonal to \mathbf{E} and \mathbf{k} .

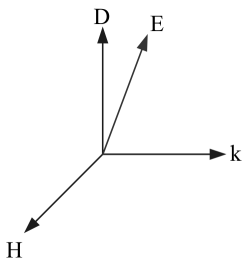
Plane waves in anisotropic crystals

- Exercise: Prove that \mathbf{D} , \mathbf{H} and \mathbf{k} are mutually orthogonal to each other

Plane waves in anisotropic crystals

- Exercise: Prove that \mathbf{D} , \mathbf{H} and \mathbf{k} are mutually orthogonal to each other

We can easily prove it from the divergence equations $\nabla \cdot \mathbf{d} = \nabla \cdot \mathbf{b} = 0$, which lead to $\mathbf{k} \cdot \mathbf{D} = \mathbf{k} \cdot \mathbf{H} = 0$.



- Advanced Question: What are the physical implications of this result?

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- A. Yariv, *Photonics: Optical Electronics in Modern Communications* (Oxford University Press, 2006). Chapter 1
- A. Yariv, *Quantum electronics* (Wiley, 1989). Chapter 5