EE391D Advanced Topics in Photonics

Light propagation in anisotropic crystals
lesson 3

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Outline

Uniaxial crystals

$$\underline{\underline{\epsilon}} = \epsilon_0 \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \tag{1}$$

To calculate the plane wave solutions that can propagate in this material, we begin from the index equation:

$$\frac{s_x^2}{n^2 - n_1^2} + \frac{s_y^2}{n^2 - n_1^2} + \frac{s_z^2}{n^2 - n_3^2} = \frac{1}{n^2} \tag{2}$$

with $n_i = \sqrt{\epsilon_i}$. This equation can be rewritten in the following form:

$$\left(n^2 - n_1^2\right) \left[n_3^2 n_1^2 - n^2 \left(n_1^2 s_x^2 + n_1^2 s_y^2 + n_3^2 s_z^2\right)\right] = 0, \tag{3}$$

which admits the following solutions:

Uniaxial crystals

- $n = n_1$. This is an ordinary wave "o". From the field equation:

$$\left(n^2 - n_k^2\right) E_k = n^2 s_k(\hat{s} \cdot \mathbf{E}), \tag{4}$$

with k=1,2,3. we have:

I For $k = 1$ or $k = 2$, we have $n^2 - n_k^2 = n^2 - n_1^2 = n_1^2 - n_1^2 = 0$, which substituted into (4) leads to

$$\hat{s} \cdot \mathbf{E} = 0, \tag{5}$$

which implies $\mathbf{E} \perp \mathbf{k}$

II For $k = 3$, we have conversely:

$$\left(n_1^2 - n_3^2\right) E_z = n^2 s_z(\hat{s} \cdot \mathbf{E}) = 0, \tag{6}$$

which leads to $E_z = 0$.

This is a classical ordinary wave with electric field $\mathbf{E}$ lying in the plane defined by the axis 1 and 2, and wavevector $\mathbf{k} = \frac{\omega}{c} n_1 \hat{s}$ perpendicular to $\mathbf{E}$.

Uniaxial crystals

- $n = \frac{n_1 n_3}{\sqrt{n_1^2(s_x^2+s_y^2)+n_3 s_z^2}}$. This case yields an extraordinary wave "e". If the wavevector $\mathbf{k} = k[\sin\theta, 0, \cos\theta]$ lies in the plane (x, z), we have:

$$n(\theta) = \frac{n_1 n_3}{\sqrt{n_1^2 \sin\theta^2 + n_3^2 \cos\theta^2}}, \tag{7}$$

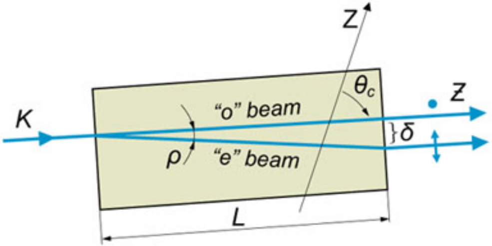
as obtained in the previous lesson. From the field equation (4), we then have the expression of the electric field of the plane wave:

$$E_k = \frac{n^2 s_k \left(\hat{s} \cdot \mathbf{E}\right)}{n^2 - n_k^2}. \tag{8}$$

Birefringence is observed in the refractive index $n(\theta)$, which changes with the direction of **k**.

Uniaxial crystals

In a Uniaxial crystal, for a given frequency and wavevector **k**, we therefore have 2 plane wave solutions: one represented by an ordinary wave, and another represented by an extraordinary wave.



Advanced question: in the case of localized beams given by the superposition of collimated plane waves, such as Gaussian beams, what are the physical implications of this result?

Outline

The index ellipsoid

This is powerful graphical method that is extensively used. It is completely equivalent to the previous formulation. We begin by expressing the iso-energy surfaces, obtained from the dielectric energy density:

$$\mathcal{W}_e = \frac{1}{2}\mathbf{e}^T \cdot \underline{\underline{\epsilon}} \cdot \mathbf{e} = \frac{e_x^2\epsilon_1}{2} + \frac{e_y^2\epsilon_2}{2} + \frac{e_z^2\epsilon_3}{2}. \tag{9}$$

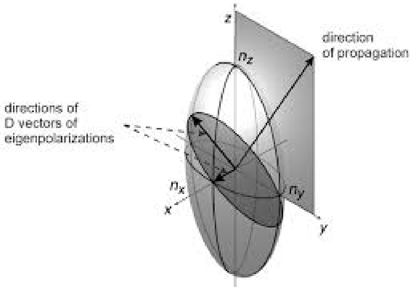
By defining the following ‘position’ vector $\mathbf{r} \equiv [x, y, z] = \frac{\mathbf{d}}{\sqrt{2\mathcal{W}_e}}$, with $\mathbf{d} = \underline{\underline{\epsilon}} \cdot \mathbf{e}$ the dielectric displacement, we obtain the index ellipsoid equation:

$$\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} = 1. \tag{10}$$

This equation, which describes an ellipsoid in the space defined by the position vector $\mathbf{r}$, can be used to find the index n and the direction of polarization of the displacement $\mathbf{d}$ of the plane waves supported by the anisotropic material.

The index ellipsoid

1. Find the intersection ellipse between a plane through the origin that is normal to the direction of wavevector $\hat{s}$ and the ellipsoid (gray area in the figure).
2. The two axes of the intersection ellipse have semi-lengths n_1 and n_2, corresponding to the index of the plane waves supported by the material.
3. The two semi-axis of the intersection ellipse are parallel to the direction of **d** of the plane waves of the anisotropic crystal.



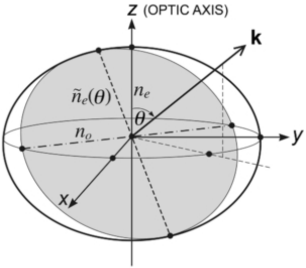
Outline

Uniaxial crystals: again

Exercise: study Uniaxial crystals with the index ellipsoid method. The index ellipsoid equation reads as follows:

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1, \tag{11}$$

with n_o and n_e representing the ordinary and the extraordinary index, respectively.

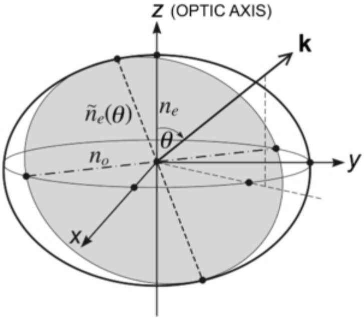


Uniaxial crystals: again

For any direction of **k**, one semi-axis of the intersection ellipse has always length $n = n_o$, corresponding to an ordinary wave, while the other is an extraordinary wave with $n = \tilde{n}_e(\theta)$:

$$\tilde{n}_e(\theta) = \frac{n_o n_e}{\sqrt{n_e^2 \sin\theta^2 + n_o^2 \cos\theta^2}} \qquad (12)$$

The refractive index associated to the extraordinary wave takes any value between n_e and n_o, depending on the direction of **k**.



Uniaxial crystals: again

Exercise: consider a Uniaxial crystal with length L along z and optic axis in the plane (x, y). A plane wave with $\mathbf{k}$ parallel to $\hat{z}$ impinges on the crystal. Calculate the emerging Electric field at the output of the crystal.

Outline

Reference texts

- A. Yariv, *Photonics: Optical Electronics in Modern Communications* (Oxford University Press, 2006). Chapter 1
- A. Yariv, *Quantum electronics* (Wiley, 1989). Chapter 5