

# EE391D Advanced Topics in Photonics

Light propagation in anisotropic crystals

lesson 4

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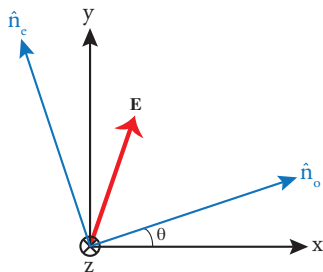
February 2, 2016

# Outline

- 1 Propagation of light in Uniaxial crystals
- 2 Applications to optical instruments
  - Half-wave retarder plate
  - Quarter-wave plate
- 3 Reference texts

# Propagation of light in Uniaxial crystals

Exercise: consider a Uniaxial crystal with length  $L$  along  $z$  and optic axis in the plane  $(x, y)$ . A plane wave with  $\mathbf{k}$  parallel to  $\hat{z}$  impinges on the crystal. Calculate the emerging Electric field at the output of the crystal.



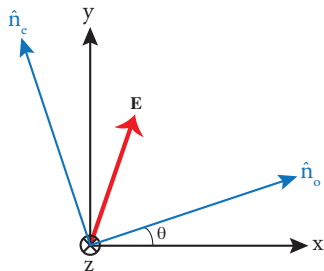
$\hat{n}_o$  and  $\hat{n}_e$  are unit vectors along the principal ordinary and extraordinary axes, respectively, while  $\mathbf{E} = [E_x, E_y, 0]$  is electric input field.

# Propagation of light in Uniaxial crystals

For  $\mathbf{k}$  parallel to  $z$ , the plane waves propagating in the uniaxial crystals have refractive index  $n_e$  and  $n_o$ , with polarization vectors parallel to  $\hat{n}_e$  and  $\hat{n}_o$ , respectively. To solve for the propagation of the electric field, we decompose the field along  $\hat{n}_e$  and  $\hat{n}_o$ :

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} E_o \\ E_e \end{bmatrix} \quad (1)$$

Each component  $E_e$  and  $E_o$  represents a plane wave solution inside the anisotropic crystal propagating with wavevector  $\mathbf{k}_e = \frac{\omega}{c} n_e \hat{z}$  and  $\mathbf{k}_o = \frac{\omega}{c} n_o \hat{z}$ , respectively.



# Propagation of light in Uniaxial crystals

After a distance  $z = L$  in the uniaxial crystal, plane waves propagate as follows:

$$\begin{aligned} E_o(L) &= E_o(0)e^{-ik_{oz}L} = E_o e^{-i\frac{\omega}{c}n_oL} \\ E_e(L) &= E_e(0)e^{-ik_{ez}L} = E_e e^{-i\frac{\omega}{c}n_eL} \end{aligned} \quad (2)$$

In matrix form, the last expression reads:

$$\begin{bmatrix} E_o(L) \\ E_e(L) \end{bmatrix} = e^{-i\frac{\omega}{c}L \cdot \underline{\underline{n}}} \begin{bmatrix} E_o(0) \\ E_e(0) \end{bmatrix}, \quad \underline{\underline{n}} = \begin{bmatrix} n_o & 0 \\ 0 & n_e \end{bmatrix} \quad (3)$$

Due to birefringence, the two plane waves, originally having the same phase, acquire a phase delay  $\Delta\phi = (n_o - n_e)\frac{\omega L}{c}$  at  $z = L$ .

## Propagation of light in Uniaxial crystals

It is convenient to write the propagation of the field as a function of the phase delay  $\Delta\phi$ :

$$\begin{bmatrix} E_o(L) \\ E_e(L) \end{bmatrix} = e^{-i\frac{1}{2}(n_e+n_o)\frac{\omega}{c}L} \begin{bmatrix} e^{-i\frac{\Delta\phi}{2}} & 0 \\ 0 & e^{i\frac{\Delta\phi}{2}} \end{bmatrix} \begin{bmatrix} E_o(0) \\ E_e(0) \end{bmatrix} = \underline{\underline{W}} \cdot \begin{bmatrix} E_o(0) \\ E_e(0) \end{bmatrix} \quad (4)$$

At the output of the crystal, in order to get back to the original coordinates, we apply a rotation:

$$\begin{bmatrix} E_x(L) \\ E_y(L) \end{bmatrix} = \underline{\underline{R}} \cdot \underline{\underline{W}} \cdot \underline{\underline{R}}^\dagger \cdot \begin{bmatrix} E_x(0) \\ E_y(0) \end{bmatrix}, \quad (5)$$

with:

$$\underline{\underline{R}} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad (6)$$

defining a rotation matrix in the  $(x, y)$  plane.

# Propagation of light in Uniaxial crystals

The total input-output transfer matrix of the structure is then:

$$\underline{\underline{R}} \cdot \underline{\underline{W}} \cdot \underline{\underline{R}}^\dagger = \begin{bmatrix} e^{-i\frac{\Delta\phi}{2}} \cos \theta^2 + e^{i\frac{\Delta\phi}{2}} \sin \theta^2 & -i \sin \frac{\Delta\phi}{2} \sin 2\theta \\ -i \sin \frac{\Delta\phi}{2} \sin 2\theta & e^{-i\frac{\Delta\phi}{2}} \sin \theta^2 + e^{i\frac{\Delta\phi}{2}} \cos \theta^2 \end{bmatrix} \quad (7)$$

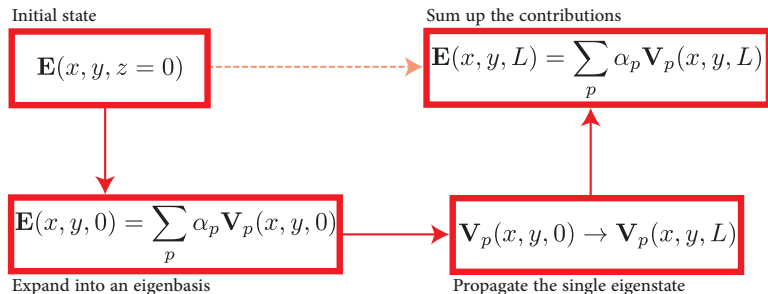
The approach used above to model the propagation of electric field in anisotropic crystals is known as **Jones Calculus**, while the  $2 \times 2$  transfer matrices that we used in the previous equations are known as **Jones Matrices**.

Exercise: verify that the transfer matrix is unitary, i.e.,  $\underline{\underline{T}} \cdot \underline{\underline{T}}^\dagger = \underline{\underline{1}}$ , with  $\underline{\underline{1}}$  being the identity matrix and  $\underline{\underline{T}} = \underline{\underline{R}} \cdot \underline{\underline{W}} \cdot \underline{\underline{R}}^\dagger$ .

Advanced question: what is the physical interpretation of this result?

# Propagation of light in Uniaxial crystals

From a more general perspective, the approach we used to solve the propagation problem is based on eigenvalue decomposition:



and exploits the linearity of Maxwell equations. For nonlinear materials, this approach cannot be used and other strategies are employed based on more complex transforms.



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## Half-wave retarder plate

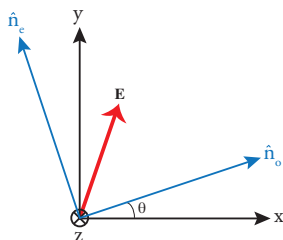
It is composed by a uniaxial crystal in the configuration studied in the previous exercise, with:

$$\Delta\phi = (n_e - n_o) \frac{\omega}{c} L = \pi, \quad (8)$$

The length of the crystal is then  $L = \frac{\lambda}{2\Delta n}$ , with  $\Delta n = n_e - n_o$ . In this system, when  $\theta = \frac{\pi}{4}$ , the transfer matrix  $\underline{\underline{T}}$  reads:

$$\underline{\underline{T}} = -i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -i\sigma_1, \quad (9)$$

being  $\sigma_1$  the Pauli matrix.

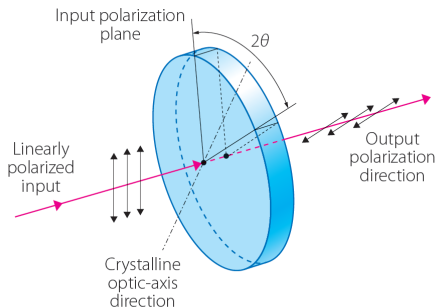


## Half-wave retarder plate

An x-polarized electric field at the beginning  $\mathbf{E} = [1, 0]$ , transforms into:

$$\mathbf{E}(L) = -i\sigma_1\mathbf{E} = -i \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (10)$$

which represents a y-polarized field at the output.



Performance of half ( $\lambda/2$ ) waveplate

# Half-wave retarder plate

Exercise: calculate the evolution of a circular polarized electric field

$$\mathbf{E} = \frac{1}{\sqrt{2}}[1, i]$$

Advanced question: design a power controller by using an half-wave retarder.

# Half-wave retarder plate

Exercise: calculate the evolution of a circular polarized electric field

$$\underline{\mathbf{E}} = \frac{1}{\sqrt{2}}[1, i]$$

**Advanced question:** design a power controller by using an half-wave retarder. We can use a half-wave retarder plate followed by a polarizer. We can verify that the transfer matrix of a polarizer is:

$$\underline{\underline{P}}_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{\underline{P}}_y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (11)$$

for an  $x$ -polarizer and  $y$ -polarizer, respectively. If we orient the half-wave retarder plate at a generic  $\theta$ , the transfer matrix becomes:

$$\underline{\underline{T}} = -i \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \quad (12)$$

## Half-wave retarder plate

Consider a generic field at the input linearly polarized, in this example along  $x$ ,  $\mathbf{E} = [E_x, 0]$ . At the output of the system crystal + polarizer, we have:

$$\begin{bmatrix} E_x(L) \\ E_y(L) \end{bmatrix} = \underline{\underline{P}}_x \cdot \underline{\underline{T}} \cdot \begin{bmatrix} E_x \\ 0 \end{bmatrix} = \begin{bmatrix} E_x \cos 2\theta \\ 0 \end{bmatrix}, \quad (13)$$

as a result, the output intensity  $I = |\mathbf{E}|^2 = |E_x|^2 \cos^2 2\theta$  can be modulated between 0 and  $|E_x|^2$  by simply acting on the rotation angle of the crystal. When  $\theta = \frac{\pi}{4}$ , the output intensity is 0, as expected. In this condition, in fact, the half-wave retarder rotates the input polarization and the polarizer prevents the beam to pass through the system.

## Quarter-wave retarder plate

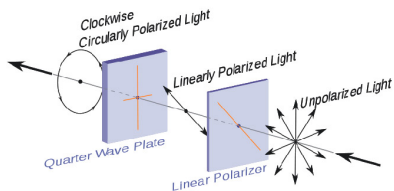
It is composed by a uniaxial crystal with:

$$\Delta\phi = (n_e - n_o)\frac{\omega}{c}L = \frac{\pi}{2}, \quad (14)$$

The length of the crystal is then  $L = \frac{\lambda}{4\Delta n}$ , with  $\Delta n = n_e - n_o$ . In this system, when  $\theta = \frac{\pi}{4}$ , the transfer matrix  $\underline{T}$  reads:

$$\underline{T} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \quad (15)$$

This transforms linearly polarized beams into circularly polarized light and vice-versa.



## Retarder plates

Advanced question: A student is investigating the propagation of light in a system composed by two polarizers in cross configuration. Due to this configuration, no light is expected to emerge from the system. However, the student observes that if he puts a half-retarder plate in between the two polarizers, some light emerges from the system. Explain the phenomenon.



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## Reference texts

- A. Yariv, *Photonics: Optical Electronics in Modern Communications* (Oxford University Press, 2006). Chapter 1
- A. Yariv, *Quantum electronics* (Wiley, 1989). Chapter 5