EE391D Advanced Topics in Photonics

Waveguide theory
lesson 9

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Outline

Mode normalization and orthogonality

Advanced question: why the factor $\frac{1}{4}$ in the orthogonality condition and what is the physical meaning of Eq. 6 of lesson 8?

From the definition of the Poynting vector:

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{E} \times \mathbf{H}\right), \tag{1}$$

the power P carried by the $\mu-$th mode can be defined as follows:

$$P = \iint \mathbf{S} \cdot \hat{z} dxdy = \frac{1}{2}\Re\left\{\iint \left(\mathbf{E}_\mu \times \mathbf{H}_\mu^*\right) \cdot \hat{z} dxdy\right\}, \tag{2}$$

which is equivalent to:

$$P = \frac{1}{4}\iint \left(\mathbf{E}_\mu \times \mathbf{H}_\mu^* + \mathbf{E}_\mu^* \times \mathbf{H}_\mu\right) \cdot \hat{z} dxdy. \tag{3}$$

Mode normalization and orthogonality

Therefore the mode normalization condition:

$$\frac{1}{4}\iint \left(\mathbf{E}_\mu \times \mathbf{H}_\nu^* + \mathbf{E}_\mu^* \times \mathbf{H}_\nu\right) \cdot \hat{z}\, dxdy = \delta_{\nu\mu}, \tag{4}$$

implies that each mode carried a unitary power along z. With this normalization, the power P carried by a superposition of modes:

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \sum_n c_n \begin{bmatrix} \mathbf{E}_n \\ \mathbf{H}_n \end{bmatrix} e^{-i\beta_n z}, \tag{5}$$

is simply:

$$P = \frac{1}{4}\iint (\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}) \cdot \hat{z}\, dxdy = \sum_n |c_n|^2 \tag{6}$$

Outline

Mode completeness

The set of waveguide modes (guided + radiation):

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \sum_n c_n \begin{bmatrix} \mathbf{E}_n \\ \mathbf{H}_n \end{bmatrix} e^{-i\beta_n z}, \tag{7}$$

forms a complete set. This implies that any solution can be expressed as a combination of modes. In order to understand from a more physical perspective the meaning of completeness, we consider the following problem.

Mode completeness

Exercise: A generic electromagnetic wave impinges on a waveguide at $z = 0$ (butt-coupling). Calculate how much energy is coupled to each mode and propagate the field inside the waveguide.

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We solve the problem from a very general perspective, which is called normal mode decomposition and is a general technique to solve wave propagation problems.

Once the beam reaches the waveguide, part of it is reflected and part is transmitted. We denote the transmitted part with the following 'vector':

$$\psi = \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} \tag{8}$$

The transmitted part propagates inside the waveguide. In this medium, each propagating field is either a mode or a combination of modes. This allows to expand the transmitted field as follows:

$$\psi = \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \sum_n c_n \psi_n, \tag{9}$$

Mode completeness

with ψ_n a mode of the structure:

$$\psi_n = \begin{bmatrix} \mathbf{E}_n \\ \mathbf{H}_n \end{bmatrix} e^{-i\beta_n z}. \tag{10}$$

The index n in the sum appearing in (11) is intended discrete for guided modes, and continuos for radiation modes, i.e.,:

$$\psi = \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \sum_{\nu} c_\nu \begin{bmatrix} \mathbf{E}_\nu \\ \mathbf{H}_\nu \end{bmatrix} e^{-i\beta_\nu z} + \int d\nu c(\nu) \begin{bmatrix} \mathbf{E}(\nu) \\ \mathbf{H}(\nu) \end{bmatrix} e^{-i\beta(\nu) z}, \tag{11}$$

Mode completeness

In order to calculate the coefficients c_n appearing in (11), we use the orthogonality condition of the modes:

$$\langle \psi_n, \psi_m \rangle = \delta_{nm}, \tag{12}$$

where the operator $\langle ... \rangle$ defines the orthogonality condition (4). By then multiplying (11) by $\langle \psi_m,$ we obtain:

$$\langle \psi_m, \psi \rangle = \sum_n c_n \langle \psi_n, \psi_m \rangle = c_m, \tag{13}$$

which allows to immediately calculate the coefficients in the expansion through the metric defined by the mode power:

$$c_n = \langle \psi_n, \psi \rangle = \frac{1}{4} \iint (\mathbf{E}_n \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}_n) \cdot \hat{z} dxdy. \tag{14}$$

The integral appearing in the expression of c_n is called overlap integral between the mode and the input field.

Mode completeness

Question: how to excite a specific mode and what are the physical implications of (14)?

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The overlap integral varies between 0 (in the case where the input field and the mode are orthogonal to each other), and 1 (when the input field is exactly a mode). This implies that in order to excite a specific mode in butt-coupling, we need to physically match the spatial profile of the mode with the input profile launched in the waveguide.

Outline

Planar waveguides

Planar waveguides are waveguides structures symmetric along a transverse direction (e.g., y, which implies $\partial_y = 0$).

Exercise: starting from Maxwell equations demonstrates that in this particular situation waveguide modes can be decoupled into two distinct and orthogonal sets: TE modes (E_y, H_x, H_z) and TM modes (H_y, E_x, E_z).

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$$TE : \begin{cases} i\beta_\nu E_y = -i\omega\nu H_x, \\ \partial_x E_y = -i\omega\nu H_z, \\ \partial_x H_z + i\beta_\nu H_x = -i\omega\epsilon E_y \end{cases} \tag{15}$$

$$TM : \begin{cases} i\beta_\nu H_y = i\omega\epsilon E_x, \\ \partial_x H_y = i\omega\epsilon E_z, \\ \partial_x E_z + i\beta_\nu E_x = i\omega\mu H_y, \end{cases} \tag{16}$$

with $\epsilon = \epsilon_0 n(x)^2$

Planar waveguides

By solving for a single field in both TE and TM, we obtain the following equations for the calculation of the modes:

$$\begin{aligned} TE: & \qquad \partial_x^2 E_y = \left(\beta_\nu^2 - n^2 k_0^2\right) E_y, \\ TM: & \qquad n^2 \partial_x \left(\frac{1}{n^2}\partial_x H_y\right) = \left(\beta_\nu^2 - n^2 k_0^2\right) H_y, \end{aligned} \tag{17}$$

with $k_0 = \frac{2\pi}{\lambda}$. From (4), the orthogonality condition read as follows:

$$\delta_{\nu\mu} = \begin{cases} -\frac{1}{2}\int E_{y\nu} H_{x\mu}^* dx = \frac{1}{2}\frac{\beta_\nu}{\omega\mu}\int E_{y\nu} E_{y\mu}^* dx & (TE) \\ -\frac{1}{2}\int H_{y\nu}^* E_{x\mu} dx = \frac{1}{2}\frac{\beta_\nu}{\omega\epsilon_0}\int \frac{1}{n^2} H_{y\nu} H_{y\mu}^* dx & (TM) \end{cases} \tag{18}$$

Outline

Reference texts

- H. Nishiara, *Optical Integrated Circuits* (McGraw Hill, 1989). Chapters 2 & 3.
- K. Okamoto, *Fundamentals of Optical Waveguides* (Academic Press, 2010). Chapter 1.
- C. A. Balanis, *Advanced Engineering Electromagnetics* (Wiley, 1989). Chapter 7.