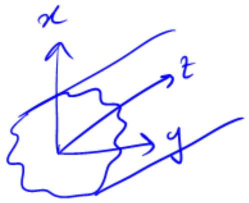
Lesson 12 → 24-MARCH-2012

Coupled Mode theory in space.

The coupled mode theory is an exact formalism for studying the propagation of modes in waveguides characterized by perturbations of arbitrary form.

We begin by considering a general 2D waveguide:



$$\underline{e} = \frac{1}{2}\underline{E}\, e^{j\omega t} + c.c.$$

$$\underline{h} = \frac{1}{2}\underline{H}\, e^{j\omega t} + c.c.$$

we decompose the fields into a Transverse and a longitudinal component:

$$\begin{cases} \underline{E} = \underline{E}_T + \underline{E}_z \\ \underline{H} = \underline{H}_T + \underline{H}_z \end{cases}$$

we then expand the dielectric permittivity $\varepsilon(x,y)$ in the following form:

$$\varepsilon(x,y) = \underbrace{\varepsilon'(x,y)}_{\text{DEFINES THE WAVEGUIDE}} + \overbrace{\Delta\varepsilon(x,y,z)}^{\text{ARBITRARY PERTURBATION}}$$

Maxwell's equations describe the propagation of ligth in the system

$$\begin{cases} \nabla \times \underline{E} = -j\omega\mu_0 \underline{H} \\ \nabla \times \underline{H} = j\omega\varepsilon_0(\varepsilon' + \Delta\varepsilon)\underline{E} \end{cases} ; \quad \begin{cases} \underline{E} = \underline{E}_T + \underline{H}_z \\ \underline{H} = \underline{H}_T + \underline{H}_z \end{cases}$$

while if we consider the waveguide only:

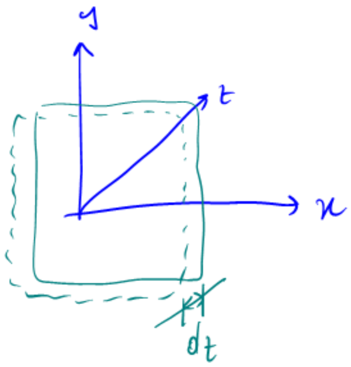
$$\begin{cases} \nabla \times \underline{E}^{(w)} = -j\omega\mu\, \underline{H}^{(w)} \\ \nabla \times \underline{H}^{(w)} = j\omega\varepsilon_0\varepsilon'\underline{E}^{(w)} \end{cases} ; \quad \begin{cases} \underline{E}^{(w)} = \underline{E}_T^{(w)} + \underline{E}_z^{(w)} \\ \underline{H}^{(w)} = \underline{H}_T^{(w)} + \underline{H}_z^{(w)} \end{cases}$$

we will obtain in general two different solutions, labeled $\underline{E}^{(w)}$ and $\underline{H}^{(w)}$.

Using the Reciprocity theorem of classical electrodynamic, we can easily demonstrate the following relationship among the two sets of solutions:

$$\boxed{\nabla\cdot\left(\underline{E}\times\underline{H}^{(\omega)*}+\underline{E}^{(\omega)*}\times\underline{H}\right)+j\omega\varepsilon_0\Delta\varepsilon\,\underline{E}^{(\omega)*}\cdot\underline{E}=\emptyset}$$

which can be verified by direct substitution.
We then integrate this relationship in a volume having infinitely large area parallel to the xy plane, and infinitely small thickness along z.



We obtain:

$$\boxed{\iint_{-\infty}^{+\infty}\partial_z\left[\underline{E}_T\times\underline{H}_T^{(\omega)*}+\underline{E}_T^{(\omega)*}\times\underline{H}_T\right]\cdot\hat{z}\,dx\,dy+j\omega\varepsilon_0\iint_{-\infty}^{+\infty}\underline{E}^{(\omega)*}\Delta\varepsilon\,\underline{E}\,dx\,dy}\qquad(11.1)$$

It is worthwhile remarking that this relation is exact and does not involve any approximation. We then expand the electric and magnetic fields $\underline{E}_T$ and $\underline{H}_T$ into guided modes (see lesson 10):

$$\begin{cases}\underline{E}_T=\sum_\nu a_\nu(z)\,\underline{\mathcal{E}}_\nu(x,y)\,e^{-j\beta_\nu z}+\int a(\nu)\,\underline{\mathcal{E}}(x,y;\nu)\,e^{-j\beta(\nu)z}\,d\nu\\ \underline{H}_T=\sum_\nu a_\nu(z)\,\underline{\mathcal{H}}_\nu(x,y)\,e^{-j\beta_\nu z}+\int a(\nu)\,\underline{\mathcal{H}}(x,y;\nu)\,e^{-j\beta(\nu)z}\,d\nu\end{cases}$$

and obtain for a generic mode μ: $\underline{E}_T^{(\omega)}=\underline{\mathcal{E}}_\mu(x,y)e^{-j\beta_\mu z}$, $\underline{H}_T^{(\omega)}=\underline{\mathcal{H}}_\mu(x,y)e^{-j\beta_\mu z}$

$$*)\quad\iint\partial_z\left[\underline{E}_T\times\underline{H}_T^{(\omega)*}+\underline{E}_T^{(\omega)*}\times\underline{H}_T\right]\cdot\hat{z}\,dx\,dy=\sum_\nu\left\{\partial_z a_\nu-j(\beta_\nu-\beta_\mu)\right\}$$

$$\times\iint\left[\underline{\mathcal{E}}_{T\nu}\times\underline{\mathcal{H}}_{T\mu}^{*}+\underline{\mathcal{E}}_{T\mu}^{*}\times\underline{\mathcal{H}}_{T\nu}\right]\cdot\hat{z}\,dx\,dy=\pm4\,\partial_z a_\mu(z)\qquad(11.2)$$

thanks to the orthonormality relation of guided modes (see lesson 10):

$$\iint [\mathcal{E}_{T\nu} \times \underline{H}^*_{T\mu} + \mathcal{E}^*_{T\mu} \times \underline{\mathcal{H}}_{T\nu}] \hat{z}\, dx\,dy = \pm 4\, \delta_{\mu\nu} \quad (\beta_\nu \geq \emptyset)$$

*) $\underline{E} = \underline{E}_T + \underline{E}_z$; From Maxwell $\begin{cases} \nabla_T \times \underline{E}_T = -j\omega\mu_0 \underline{H}_z \\ \nabla_T \times \underline{H}_T = j\omega(\varepsilon' + \Delta\varepsilon)\underline{E}_z\, \varepsilon_0 \end{cases}$

we have:

$$\underline{E} = \underline{E}_T + \frac{\nabla_T \times \underline{H}_T}{j\omega\varepsilon_0(\varepsilon' + \Delta\varepsilon)} = \int_\nu a_\nu(z) \left\{ \mathcal{E}_{T\nu} + \frac{\nabla_T \times \mathcal{H}_{t\nu}}{j\omega\varepsilon_0(\varepsilon' + \Delta\varepsilon)} \right\} e^{-j\beta_\nu z}$$

(with $\int_\nu$ denoting $\sum_\nu$ for guided modes and

we therefore have:

$$j\omega\varepsilon_0 \iint \underline{E}^{(\omega)*} \Delta\varepsilon\, \underline{E}\, dx\,dy = j\omega\varepsilon_0 \int_\nu a_\nu\, e^{-j(\beta_\nu - \beta_\mu)z}$$

$$\times \iint \mathcal{E}^*_\mu\, \Delta\varepsilon \left\{ \mathcal{E}_{T\nu} + \frac{\varepsilon'}{\varepsilon' + \Delta\varepsilon} \mathcal{E}_{z\nu} \right\} dx\,dy \quad (11.3)$$

where $\mathcal{E}_{z\nu}$ is related to the modal expansion of $\underline{E}_z$:

$$\underline{E}_z = \int_\nu a_\nu \cdot \mathcal{E}_{z\nu}\, e^{-j\beta_\nu z} \quad ; \quad \mathcal{E}_{z\nu} = \frac{\nabla \times \mathcal{H}_{T\nu}}{j\varepsilon_0\omega}$$

By substituting 11.2 and 11.3 into 11.1, we have:

$$\boxed{\pm \frac{\partial}{\partial z} a_\mu(z) = -j \int_\nu C_{\mu\nu} \cdot a_\nu(z)\, e^{-j(\beta_\nu - \beta_\mu)z}} \quad (11.4)$$

$(\beta_\mu \geq \emptyset)$

with "COUPLING COEFFICIENTS" $C_{\mu\nu}$:

$$C_{\mu\nu} = \frac{\omega\varepsilon_0}{4} \iint \mathcal{E}^*_\mu \left[\Delta\varepsilon\, \mathcal{E}_{T\nu} + \frac{\varepsilon' \Delta\varepsilon}{\varepsilon + \Delta\varepsilon} \mathcal{E}_{z\nu} \right] dx\,dy$$

which depends on the spatial overlap between the modes profile $\mathcal{E}$ and the perturbation.

it is important to stress that eq. 11.4, despite is simplicity, is completely exact. Although in many modern books the derivation of the Coupled Mode Equations (CMT) is carried out with several approximations, these are totally not necessary.