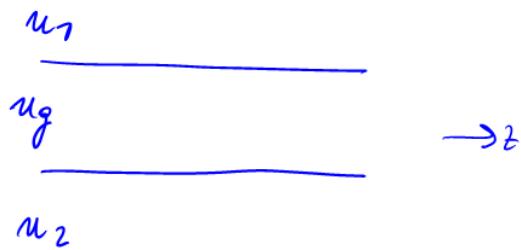


## GEOMETRICAL OPTICS ANALYSIS OF WAVEGUIDES AND RESONATORS

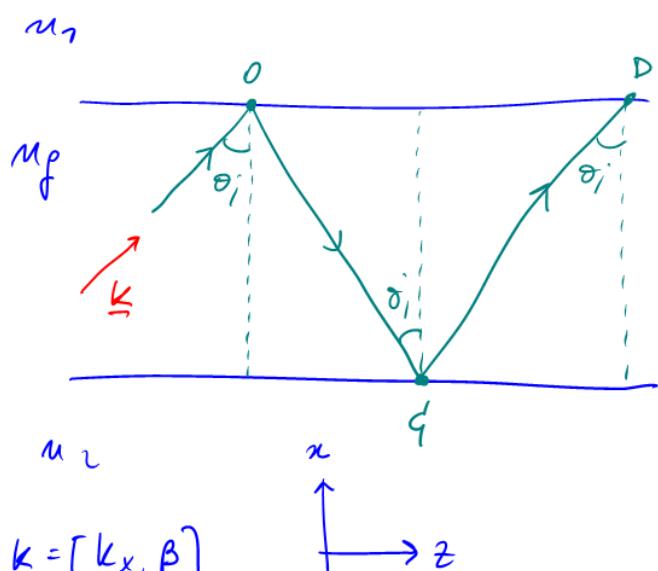
SCAB WAVEGUIDE



a field varying as  $e^{i(\beta z - \omega t)}$

Waveguides are optical systems able to support the propagation of localized radiation through modes. Modes are waves characterized by a constant intensity profile, and

propagation constant of the mode



$$\underline{k} = [k_x, \beta]$$

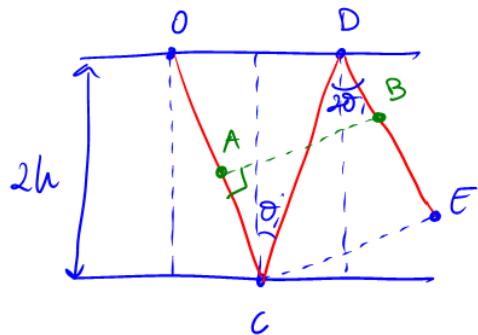
A generic ray can be trapped in the structure by total internal reflection. In this case:

$$\begin{cases} n_g \sin \theta_i > n_1 \\ n_g \sin \theta_i > n_2 \end{cases}$$

to each ray, we can associate a plane wave with a phase  $\beta \cdot z = k \cdot n_g \sin \theta_i \cdot z$ ; therefore

$$\beta = n_g k \sin \theta_i$$

in order to find the  $\delta_i$  angles (or  $\beta$ ) that lead to a guided mode, we need to impose the condition that the field propagates with a constant intensity - this can be done in the following way:



$$AB \perp OC$$

if the field has a constant intensity, the point A and B need to have the same phase.

•) EXERCISE 1: demonstrate that  $ACDB = CD + DE$

we therefore have:

$$ACDB = CD + DE = CD \cdot (1 + \cos 2\delta_i) = 2CD \cos^2 \delta_i = 4h \cos \delta_i$$

total phase shift due to distance:  $4h \cos \delta_i \cdot n_p \cdot k$

phase of plane wave:  $k \cdot z$

in order to find the total phase shift, we need to keep into account the phase shift due to the reflections at C and D. We therefore have the following condition for a mode:

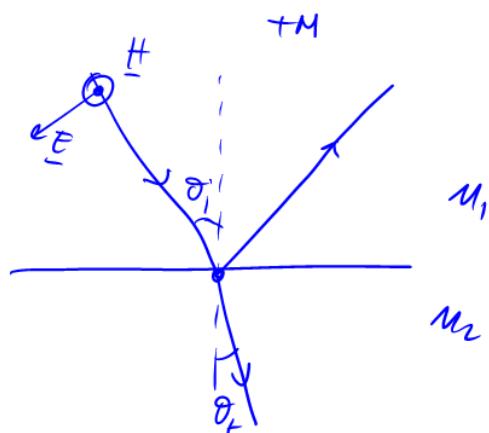
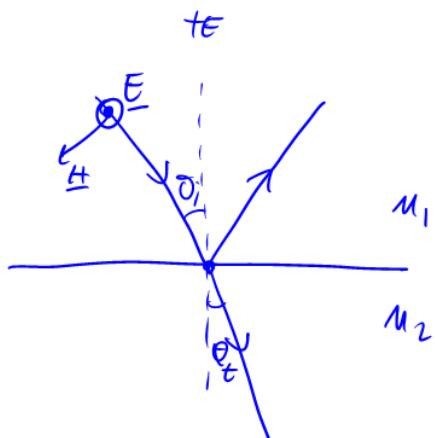
$$4n_p k h \cos \delta_i + \Phi_1 + \Phi_2 = 2m\pi \quad m=0, \pm 1, \dots$$

$\Phi_1$  and  $\Phi_2$  are the Fresnel Formulas for the reflection of plane waves at a planar interface.

$$\text{FOR TE WAVES : } \phi_j = -2 \operatorname{tg}^{-1} \left[ \frac{i}{g} \sqrt{1-g^2 - \frac{n_j^2}{n_0^2}} \right] \quad (5.1) \quad (5)$$

$$\text{FOR TM WAVES : } \phi_j = -2 \operatorname{tg}^{-1} \left[ \frac{n_0^2}{n_j^2} \frac{1}{g} \sqrt{1-g^2 - \frac{n_j^2}{n_0^2}} \right] \quad (5.2)$$

$$g = \cos \theta_i ; \quad j=1,2$$



let us derive Eqs. 5.1 & 5.2  
FRESNEL FORMULAE:

(TM)

$$\frac{E_R}{E_i} = \frac{\sin \theta_i \cdot \cos \theta_i - \sin \theta_t \cdot \cos \theta_t}{\sin \theta_i \cos \theta_i + \sin \theta_t \cdot \cos \theta_t} ; \quad \frac{E_R}{E_i} = - \frac{\sin \theta_i \cos \theta_t - \sin \theta_t \cos \theta_i}{\sin \theta_i \cos \theta_t + \sin \theta_t \cos \theta_i}$$

(TE)

for Total internal reflection:

$$\sin \theta_t = \frac{\sin \theta_i}{n} \quad n = \frac{n_2}{n_1}$$

$$\cos \theta_t = \sqrt{\frac{\sin^2 \theta_i}{n^2} - 1}$$

we therefore have :

TM

$$\frac{E_R}{E_t} = \frac{n^2 \cos \theta_i - i \sqrt{\sin^2 \theta_i - n^2}}{n^2 \cos \theta_i + i \sqrt{\sin^2 \theta_i - n^2}} = R_{TM} ; \quad \frac{E_R}{E_t} = \frac{\cos \theta_i - i \sqrt{\sin^2 \theta_i - n^2}}{\cos \theta_i + i \sqrt{\sin^2 \theta_i - n^2}} = R_{TE}$$

TE

(6)

we can write:

$$R_{TM} = z \cdot (z^*)^{-1} ; z = n^2 \cos \theta_i - j \sqrt{\sin \theta_i^2 - n^2}$$

$$R_{TE} = q \cdot (q^*)^{-1} ; q = \cos \theta_i - j \sqrt{\sin \theta_i^2 - n^2}$$

and hence

$$R_{TM} = e^{i\delta_{TM}} = e^{i2\alpha_{TM}} , \text{ with } z = a e^{i\alpha_{TM}}$$

$$R_{TE} = e^{i\delta_{TE}} = e^{i2\alpha_{TE}} , \text{ with } q = b e^{i\alpha_{TE}}$$

we therefore have:

$$\begin{aligned} \delta_{TM} &= 2\alpha_{TM} = 2 \cdot \operatorname{tg}^{-1} \left[ - \frac{\sqrt{\sin \theta_i^2 - n^2}}{n^2 \cos \theta_i} \right] = \\ &= -2 \operatorname{tg}^{-1} \left[ \frac{1}{n^2 g} \cdot \sqrt{1 - g^2 - n^2} \right] \end{aligned}$$

$$\begin{aligned} \delta_{TE} &= 2\alpha_{TE} = -2 \operatorname{tg}^{-1} \left[ \frac{\sqrt{\sin \theta_i^2 - n^2}}{\cos \theta_i} \right] = \\ &= -2 \operatorname{tg}^{-1} \left[ \frac{1}{g} \sqrt{1 - g^2 - n^2} \right] \end{aligned}$$

### SYMMETRIC WAVEGUIDE, TE MODES

We begin our analysis by considering a symmetric waveguide, with  $\mu_1 = \mu_2 \triangleq \mu$ . The dispersion relation becomes:

$$2\mu g \beta k h = m\pi + 2 \operatorname{tg}^{-1} \sqrt{\frac{\mu_g^2 - \mu^2 - \mu_g^2 g^2}{\mu_g^2 g^2}}$$

(7)

it is very convenient to express everything as a function of  $\alpha = \frac{\mu_p^2 \xi^2}{\mu_p^2 - m^2}$ , we have:

$$2k\hbar \cdot \sqrt{\mu_p^2 - m^2} \cdot \sqrt{\frac{\mu_p^2 \xi^2}{\mu_p^2 - m^2}} = m\pi + 2\log^{-1} \sqrt{1 - \frac{\frac{\mu_p^2 \xi^2}{\mu_p^2 - m^2}}{\frac{\mu_p^2 \xi^2}{\mu_p^2 - m^2}}}$$

or equivalently:

$$V \cdot \sqrt{\alpha} = m\pi + 2\log^{-1} \sqrt{\frac{1-\alpha}{\alpha}}, \text{ with } \begin{cases} \alpha = \frac{\mu_p^2 \xi^2}{\mu_p^2 - m^2} \\ V = 2k\hbar \sqrt{\mu_p^2 - m^2} \end{cases}$$

$V$  is called NORMALIZED FREQUENCY. It is proportional to  $\omega$  but  $V$  is adimensional

$d$  is another adimensional quantity whose physical meaning can be seen in the following way:

$$\begin{aligned} \text{we define } b &= 1-\alpha = 1 - \frac{\mu_p^2 \xi^2}{\mu_p^2 - m^2} = \frac{\mu_p^2 - m^2 - \mu_p^2 \xi^2}{\mu_p^2 - m^2} = \\ &= \frac{\mu_p^2 (1-\xi^2) - m^2}{\mu_p^2 - m^2} \end{aligned}$$

since  $0 < \xi^2 < 1$  &  $\mu_p > m$ ,  $b > 0$

) EXERCISE : show that  $b < 1$

we can therefore write:

$$\beta = k \cdot \mu_p \cdot \sin\theta = k \mu_p \cdot \sqrt{1-\xi^2}, \text{ and}$$

$$b = \frac{(\beta/k)^2 - m^2}{\mu_p^2 - m^2}$$

③

the quantity  $n_g \sin\theta$  acts as an index, called  
EFFECTIVE INDEX  $n_{eff} = n_g \sin\theta$ , as it yields an  
evolution along  $z$  of the type  $e^{ik n_{eff} z}$

we can therefore write:

$$b = \frac{n_{eff}^2 - n^2}{n_g^2 - n^2} \quad b \text{ acts as a NORMALIZED EQUIVALENT  
REFRACTIVE INDEX}$$

let us now consider the dispersion relation:

$$V \cdot \sqrt{1-b} = n \cdot \pi + 2 \operatorname{dg}^{-1} \sqrt{\frac{b}{1-b}}$$

UNIVERSAL DISPERSION  
CURVE FOR TE MODES

.) EXERCISE: Plot the curve  $b(V)$

.) EXERCISE: calculate, for a given  $V$ , the number  
of guided modes

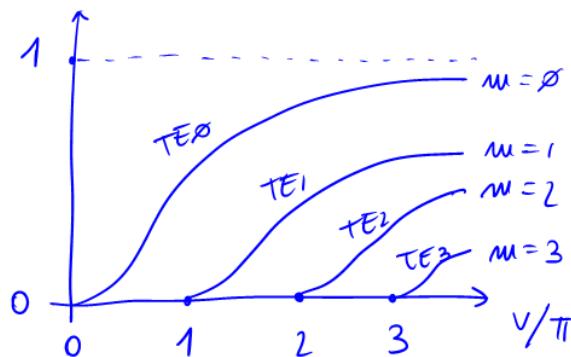
.) EXERCISE: derive the dispersion curve  
for TM modes

the result is:

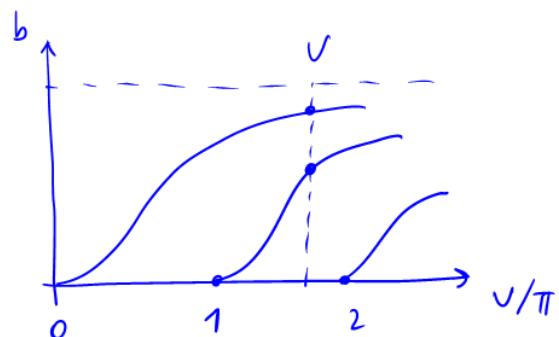
$$V \cdot \sqrt{1-b} = n \pi + 2 \operatorname{dg}^{-1} \left[ \frac{n_g^2}{n^2} \cdot \sqrt{\frac{b}{1-b}} \right]$$

Exercises on symmetric waveguides :

1)  $b(v) : v\sqrt{1-b} = m\pi + 2\operatorname{dg}^{-1}\sqrt{\frac{b}{1-b}}$



2) Number of modes for a given  $v$ .



since  $v(0) = m\pi$ , the number of modes is

$$N = \operatorname{INT}\left(\frac{v}{\pi}\right)$$

A very important physical observation is as follows - while high order modes exist above a cutoff frequency, the fundamental mode  $TE_0$  is always present regardless of the geometry of the problem.

3) TM Modes. By applying the same algebra, we end with:

$$v\sqrt{1-b} = m\pi + 2\operatorname{dg}^{-1}\left[\frac{M_0^2}{n_p^2} \sqrt{\frac{b}{1-b}}\right]$$

which is very close to the TE case, but is no longer universal as it contains the parameter  $\frac{m_p^2}{n_1^2}$ . However, in many cases  $m_p \approx n_1$ , hence, at first approximation the TE curve can be used.

### ASYMMETRIC WAVEGUIDE: TE CASE

by introducing  $\xi = \cos\theta$ , we have:

$$2m_p \xi k h = m\pi + \operatorname{dg}^{-1} \sqrt{\frac{m_p^2 - n_1^2 - m_p^2 \xi^2}{m_p^2 \xi^2}} + \operatorname{dg}^{-1} \sqrt{\frac{m_p^2 - n_1^2 + n_1^2 - n_2^2 - m_p^2 \xi^2}{m_p^2 \xi^2}}$$

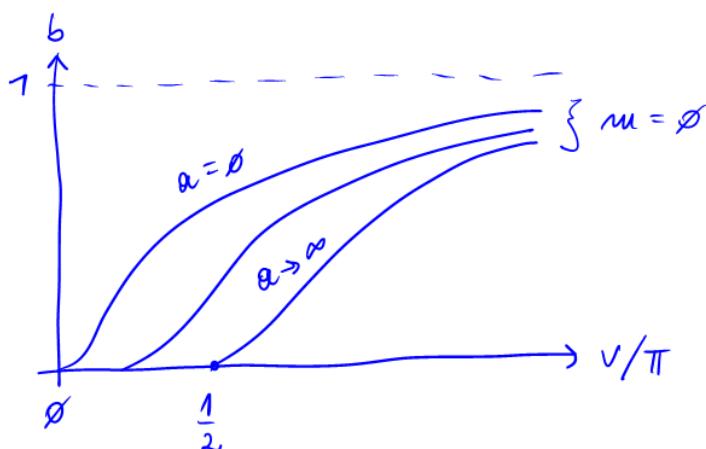
we can express both sides of the equation as a function of  $\alpha = m_p^2 \xi^2 / (m_p^2 - n_1^2)$ , as done before:

$$\sqrt{1-\alpha} = m\pi + \operatorname{dg}^{-1} \sqrt{\frac{1-\alpha}{\alpha}} + \operatorname{dg}^{-1} \sqrt{\frac{1-\alpha+\alpha}{\alpha}}$$

with  $\alpha = \frac{n_1^2 - n_2^2}{m_p^2 - n_1^2}$  playing the role of an ASYMMETRY PARAMETER

by introducing the NORMALIZED INDEX  $b = 1-\alpha$ , we have:

$$\sqrt{1-b} = m\pi + \operatorname{dg}^{-1} \sqrt{\frac{b}{1-b}} + \operatorname{dg}^{-1} \sqrt{\frac{a+b}{1-b}}, \quad m=0, 1, \dots$$



$$v(0) = m\pi + \operatorname{dg}^{-1} \sqrt{a}$$

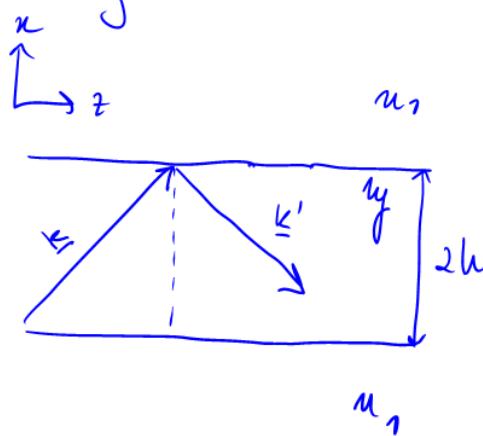
$$\lim_{a \rightarrow \infty} v(0) = \pi(m + 1/2)$$

if  $a > 0$ , or cutoff frequency exists also for TE<sub>0</sub>

(3)

## Modal profile

the last question we need to address is the profile of the guided modes. We will begin by the TE case:



inside the slab, the field is composed by two plane waves with the same  $k$ , but with opposite orientations of  $\hat{k}$  along  $x$ :

$$\text{INSIDE} \rightarrow E = E \cdot \hat{y}; E = \frac{a}{2} e^{jk_x x} + \frac{b}{2} e^{-jk_x x} = A \cos(k_x x + \varphi)$$

$$n_p^2 k^2 = k_x^2 + k_z^2 = k_x^2 + \beta^2; k_x = \sqrt{n_p^2 k^2 - \beta^2} = \alpha$$

therefore:

$$\text{INSIDE} \rightarrow E = A \cos(\alpha x + \varphi); \text{ for symmetry reasons: } \varphi = 0, \frac{\pi}{2} \Rightarrow E = \begin{cases} A \cos \alpha x \\ A \sin \alpha x \end{cases}$$

Outside the waveguide, TIR imposes the presence of evanescent fields of the type:

$$B e^{\pm j k'_x x}, \text{ with } k'_x = \sqrt{n_1 k^2 - \beta^2}, \text{ but } \beta^2 > n_1^2 k^2$$

cause  $\beta = n_{\text{eff}} \cdot k$ , hence,  $\beta^2 = n_{\text{eff}}^2 k^2 > n_1^2 k^2$

and therefore:

$$k'_x = \pm \sqrt{\beta^2 - n_1^2 k^2} = \pm \delta. \text{ the field is then:}$$

$$E = B \cdot e^{-\delta(|x|-h)}$$

SUMMARIZING:

$$E = \begin{cases} A \cos \pi x & , |x| < h \\ B e^{-\delta(|x|-h)} & , |x| > h \end{cases} ; E = \begin{cases} A \sin \pi x & , |x| < h \\ B e^{-\delta(|x|-h)} & , |x| > h \end{cases}$$

④

$\hookrightarrow B \cdot \text{sign}(x)$

) question: How to determine the value of the constants A and B?

we need to impose the continuity of the field

NOTE TE EVEN

$$A \cos \pi h = B$$

mode TE ODD

$$A \sin \pi h = B$$

we therefore have :

$$V \cdot \sqrt{1-b} = m \cdot \pi + 2 \operatorname{tg}^{-1} \sqrt{\frac{b}{1-b}}$$

$$m = 2n \text{ TE EVEN}$$

$$m = 2n+1 \text{ TE ODD}$$

$$n = 0, 1, \dots$$

$$E = \begin{cases} B \frac{\cos \pi x}{\cos \pi h} & , |x| < h \\ B \cdot e^{-\delta(|x|-h)} & , |x| > h \end{cases} ; E = \begin{cases} B \frac{\sin \pi x}{\sin \pi h} & , |x| < h \\ B \cdot \text{sign}(x) \cdot e^{-\delta(|x|-h)} & , |x| > h \end{cases}$$

TE EVEN

$$\gamma = \sqrt{\mu_g^2 k^2 - \beta^2} = \frac{2\pi}{\lambda} \sqrt{\mu_g^2 - \mu_{\text{eff}}^2}$$

TE ODD

$$\delta = \frac{2\pi}{\lambda} \sqrt{\mu_{\text{eff}}^2 - \mu_g^2}$$

) EXERCISE: write a Matlab program that, for given  $\lambda, \mu_g, \mu_{\text{eff}}, h$ , calculate the modes indices  $n_{\text{eff}}$  and plot the mode profiles.