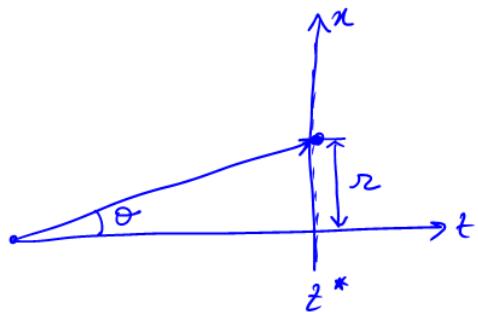


GEOMETRICAL OPTIC ANALYSIS OF CAVITIES

we begin by extending the ABCD law, introduced for Gaussian Beams, to ray optics :



a generic ray of light that propagates inside a medium can be described

by using two quantities: the slope σ and the height r calculated with respect to the vertical axis x . Assuming a paraxial approximation, we have $\sin \sigma \approx \tan \sigma \approx \sigma$.

When the ray propagates inside a generic optical system, we describe its evolution by the linear system:

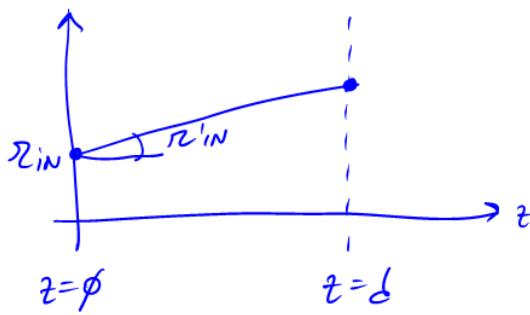
$$\begin{bmatrix} r_{\text{out}} \\ r'_{\text{out}} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_{\text{in}} \\ r'_{\text{in}} \end{bmatrix}$$

r_{in} f r'_{in} acts as a sources, while r_{out} f r'_{out} are the outcome variables

•) Exercise: calculate the ABCD matrix of a homogeneous dielectric of length d .

the ray follows a rectilinear propagation inside the material, so we have:

(2)



$$r_{\text{out}} = Ar_{\text{in}} + Br'_{\text{in}}$$

$$r'_{\text{out}} = Cr_{\text{in}} + Dr'_{\text{in}}$$

we have:

$$r'_{\text{out}} = r'_{\text{in}} \Rightarrow C=0, D=1$$

$$r_{\text{out}} = r_{\text{in}} + d \cdot r'_{\text{in}} \Rightarrow$$

$$\Rightarrow A=1, B=d$$

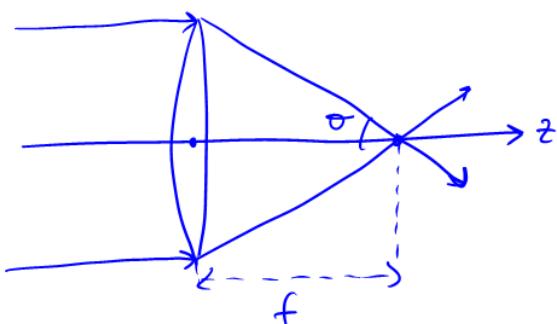
we have :

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$\text{as seen, } \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 1$$

Let us consider some common optical systems:

i) thin lens:



we can use the linear superposition principle for calculating ABCD:

at first we assume $r'_{\text{in}} = \infty$, and we get :

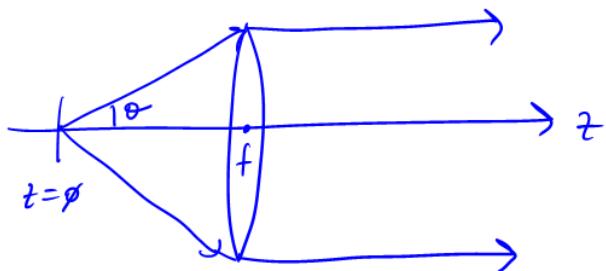
$$r_{\text{out}} = r_{\text{in}} \Rightarrow A=1$$

the output slope can be calculated from the picture and reads:

$$r'_{\text{out}} = -\theta = -\frac{r_{\text{in}}}{f} \Rightarrow C = -1/f$$

we now consider the case $r'_{\text{in}} \neq 0$

In this case is convenient to consider the reciprocal situation:



Now we have

$$R_{out} = R_{in}, \forall \theta$$

therefore $B = \emptyset$

and

$$R'_{out} = \emptyset =$$

$$= R'_{in} \cdot D + C \cdot R_{in} =$$

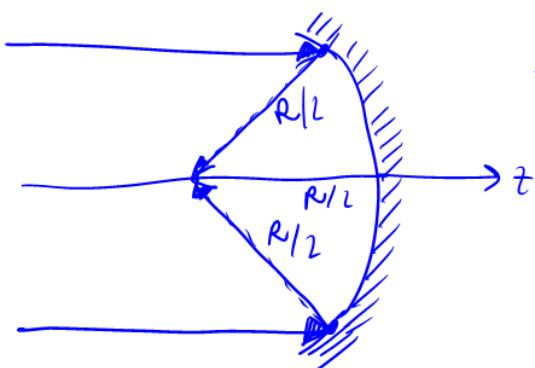
$$= R'_{in} \cdot D - \frac{R'_{in}}{f} =$$

$$= \frac{R'_{in}}{f} (D-1) = \emptyset$$

therefore $D = 1$

we have: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$ $AB - CD = \emptyset$

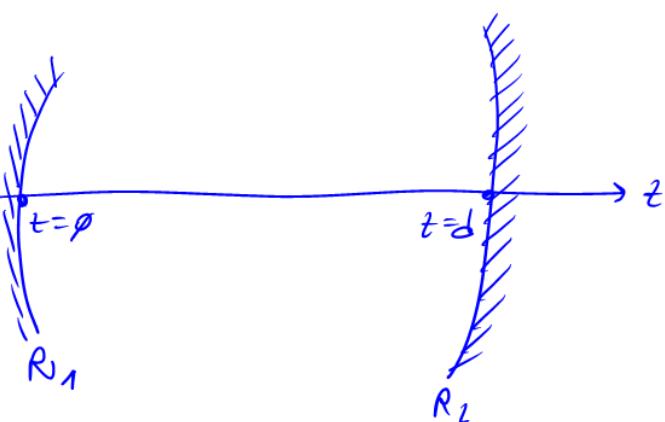
2) spherical mirror of radius R



it focuses the ray on the z axes. it is equivalent to a lens of focal $f = R/2$. the ABCD matrix of this element is :

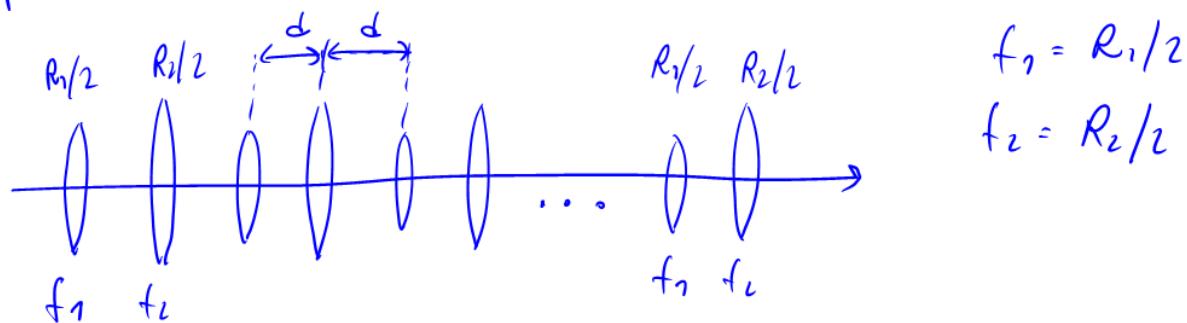
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}$$

Let us now study optical cavities formed by spherical mirrors:

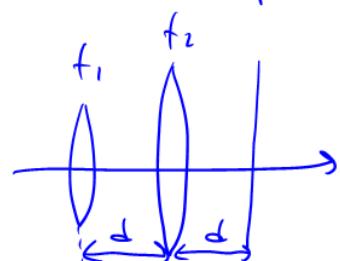


the cavity is composed by two spherical mirrors of different radius.

in order to study this system, we represent it as a periodic succession of thin lenses:



this is a periodic system, whose unit cell is:



the total ABCD matrix of the system is:

$$ABCD = ABCD_d \cdot ABCD_{f_2} \cdot ABCD_d \cdot ABCD_{f_1} =$$

$$= \begin{bmatrix} 1 - d/f_2 & d + d(1 - d/f_2) \\ -\frac{1}{f_1} - \frac{1}{f_2} \left(1 - \frac{d}{f_1}\right) & \left(1 - \frac{d}{f_1}\right) \left(1 - \frac{d}{f_2}\right) - \frac{d}{f_1} \end{bmatrix}$$

(5)

in order to analyze its properties, let us find a second order equation for the ray height r_s as it passes through the lenses. let us consider the general case:

$$\begin{bmatrix} r_{s+1} \\ r'_{s+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_s \\ r'_s \end{bmatrix} = ABCD \cdot \begin{bmatrix} r_s \\ r'_s \end{bmatrix}$$

we have:

$$r_{s+1} = A r_s + B r'_s ; \quad r'_s = \frac{1}{B} (r_{s+1} - A r_s) \quad \textcircled{A}$$

now we increment s:

$s = s+1$, from \textcircled{A}:

$$r'_{s+1} = \frac{1}{B} (r_{s+2} - A r_{s+1}) = C r_s + D r'_s \rightarrow \frac{1}{B} (r_{s+1} - A r_s)$$

we get:

$$\frac{1}{B} (r_{s+2} - A r_{s+1}) = C r_s + \frac{D}{B} (r_{s+1} - A r_s),$$

which yields:

$$r_{s+2} - (A+D) \cdot r_{s+1} + C r_s = 0$$

this is equivalent to Schrödinger equation on a lattice:

$$E \psi_m + \frac{1}{l} (\psi_{m+1} + \psi_{m-1}) = 0 \quad \text{in the tight-binding approximation}$$

these are discrete linear equations that can be solved by a normal mode decomposition or spectral analysis

we begin by introducing the translation operators: ⑥

$$\Pi r_s = r_{s+1} ; \quad \Pi^{-1} r_s = r_{s-1}$$

some important properties of Π :

$$\begin{aligned} \langle r, \Pi r' \rangle &= \sum_{n=-\infty}^{+\infty} r_n \cdot \Pi r'_n = \sum_n r_n \cdot r'_{n+1} = \\ &= \sum_n r_{n-1} \cdot r'_n = \sum_n (\Pi^* r_n) \cdot r'_n = \\ &= \langle \Pi^{-1} \cdot r_n, r_n \rangle \triangleq \langle \Pi^* r_n, r_n \rangle \end{aligned}$$

Π is a unitary operator $\Pi^{-1} = \Pi^*$, from a physical perspective, it does not change the metric of the space. $\Pi \cdot \Pi^* = \mathbb{1}$

Let us see the eigenvalues and eigenvectors of Π :

$$\Pi q_m = \beta \cdot q_m \quad q_m = q^m$$

this gives $\beta = q$ & $q_m = q^m$. for $q \in \mathbb{R}$, $q^m \rightarrow \infty$ for $m \rightarrow \infty$ and does not belong to L_2 ; so

we impose the additional condition:

$$|q|=1 \Rightarrow q = e^{i\theta}$$

we therefore have $\beta = e^{i\theta}$

Our equation can be written in the following form:

$$[\Pi + \Pi^* + (A+D)\mathbb{1}] r_s = \phi$$

we can now substitute eigenmodes of Π $r_s = e^{is\theta}$ and solve the resulting algebraic equation.

$$\left[e^{i\theta} + e^{-i\theta} + (A+D) \right] = \phi$$

$$\cos \theta = \frac{A+D}{2}$$

this equation has real solution
for θ only if $\left| \frac{A+D}{2} \right| \leq 1$

in this case $\theta \in \mathbb{R}$ and the mode is oscillating
along t : $R_s = (e^{i\theta})^s$; $|R_s| = 1$

if $\left| \frac{A+D}{2} \right| > 1$ θ becomes complex and the mode
is exponentially diverging. We therefore have:

$$\left| \frac{A+D}{2} \right| \leq 1 \rightarrow \text{STABLE CAVITY}$$

$$\left| \frac{A+D}{2} \right| > 1 \rightarrow \text{UNSTABLE CAVITY}$$

→ EXERCISE: express the stability condition as a function
of ϕ, f_1, f_2 and plot it.

STABILITY DIAGRAM OF OPTICAL RESONATORS

From the previous analysis, we demonstrated that the condition for stability in terms of cavity modes is the following:

$$-1 \leq \frac{A + D}{2} \leq 1 , \text{ or equivalently :}$$

$$0 \leq \frac{A+D+2}{4} \leq 1$$

In our case, we have:

$$\begin{bmatrix} 1-d/f_2 & d+d(1-d/f_2) \\ -1/f_1 - 1/f_2(1-d/f_1) & (1-d/f_1)(1-d/f_2) - d/f_1 \end{bmatrix}; \quad A = 1-d/f_2 \\ D = (1-d/f_1)(1-d/f_2) - d/f_1$$

$$\frac{A+D+2}{4} = \frac{1}{4} \left[1 - \frac{d}{f_2} + \left(1 - \frac{d}{f_1} \right) \left(1 - \frac{d}{f_2} \right) - \frac{d}{f_1} + 2 \right] = \\ = \left(1 - \frac{d}{2f_1} \right) \left(1 - \frac{d}{2f_2} \right)$$

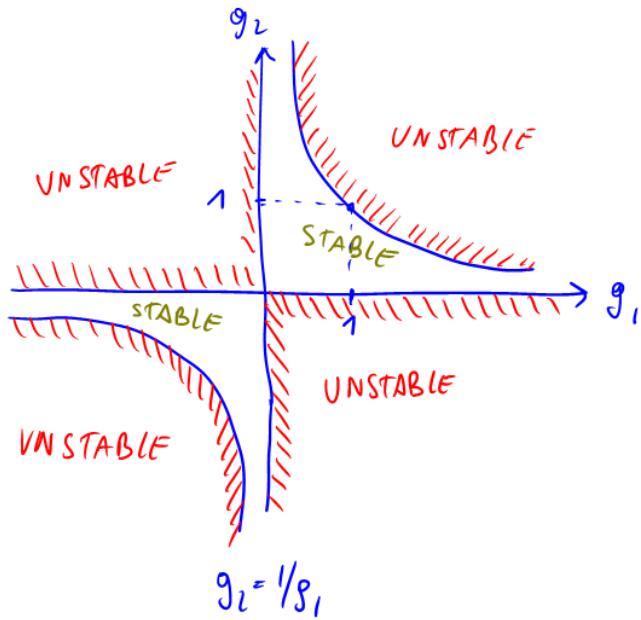
$$\begin{cases} 2f_1 = R_1 \\ 2f_2 = R_2 \end{cases} \quad \left\{ \text{radius of curvature of the spherical mirrors} \right.$$

The stability condition can be written in close form:

$$0 \leq g_1 \cdot g_2 \leq 1 \quad g_i = 1 - \frac{d}{R_i} \quad i=1,2$$

this relation is illustrated in the plot:

③



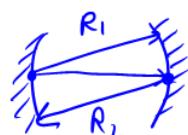
SPECIAL CASES

a) $g_1 = g_2 = 1 \Rightarrow R_i \rightarrow \infty$



plane mirrors. This system is stable

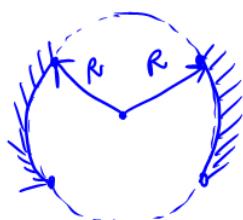
b) $g_1 = g_2 = \phi \Rightarrow R_i = d$



confocal system. this is stable

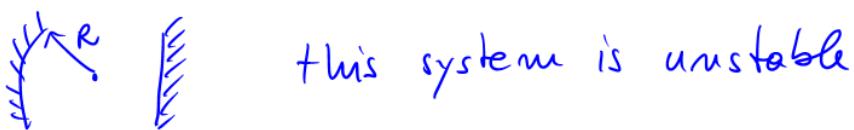
c) $g_1 = g_2 = -1 \Rightarrow R_i = d/2$

spherical system. this is stable



d) $g_1 = -1; g_2 = 1$

combination of spherical and plane

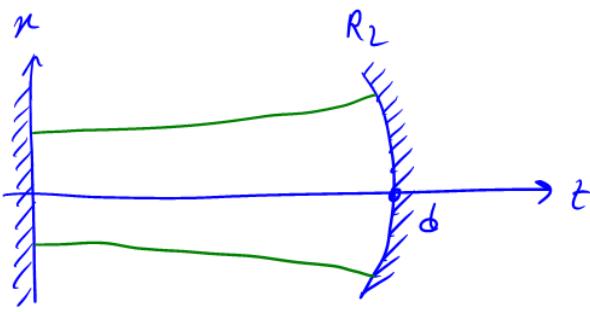


this system is unstable

GAUSSIAN BEAMS IN STABLE RESONATORS

We want to study the propagation of Gaussian Beams in a stable resonator.

We consider a simple system, made by a stable resonator with $g_1 = 1, |g_2| < 1$:



(3)

we want to calculate modes of this system. A mode, in a cavity, is characterised by an electromagnetic solution with a constant intensity in time, whose field evolves in t as $\propto e^{i\omega t}$

ω is in general complex : $\omega = \omega_0 + j\frac{1}{\tau}$

\uparrow τ decaying constant
 oscillation
 frequency

in order to calculate the Gaussian modes of this system, we use the following approach. We look for the Gaussian beams that, after reflection, reproduce themselves at the input - the solutions found in this way satisfy the definition of the mode.

let us consider a gaussian beam starting at $z=0$ with $R(0) = +\infty$ and ω_0 . the Rayleigh distance of the Beam is $L = \frac{\pi n \cdot \omega_0^2}{\lambda}$

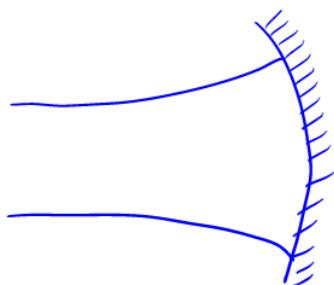
At $z=d$, the radius of curvature of the beam is :

$$R(d) = d \left[1 + \left(\frac{L_0}{d} \right)^2 \right]$$

in order to get a self-consistent solution to the problem, we can employ reciprocity (or time reversibility)

if a beam with $R(d), w_0$ is launched at $z=d$ and gets totally reflected, it will be refocused to $R(0)=\infty$ and w_0 . In order to get totally reflected, the mirror needs to match the Gaussian beam curvature in $z=d$:

$$R(d) = R_2 = d \left[1 + \left(\frac{L}{d} \right)^2 \right] \quad ①$$



we can now solve Eq. ①:

$$L = \frac{\pi n w_0^2}{\lambda} = \sqrt{d R_2} \sqrt{1 - d/R_2}$$

this solution express the minimum spot size w_0 as a function of the geometry. We then have all the parameters to characterize the Gaussian Beam, as we know

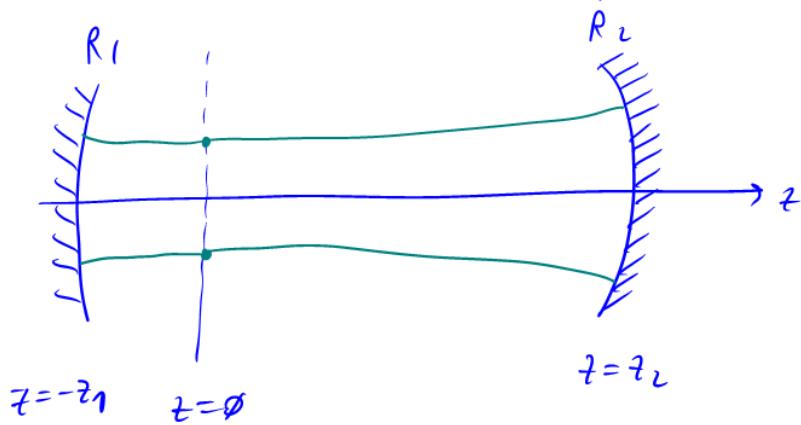
$$\begin{cases} R(0) = \infty \\ L = \frac{\pi n w_0^2}{\lambda} = \sqrt{d R_2} \sqrt{1 - d/R_2} \end{cases}$$

• exercise: calculate the expression of the waist $w(d)$ of this Gaussian mode.

$$w(d) = \frac{\sqrt{d R_2}}{\sqrt{1 - d/R_2}}$$

⑤

- Exercise : solve the generic case with two minors with $|g_1|, |g_2| < 1$



$$d = z_1 + z_2 \quad a)$$

self-consistency conditions:

$$\begin{cases} R(z_2) = z_2 \left[1 + \left(\frac{L}{z_2} \right)^2 \right] = R_2 \quad b) \\ R(z_1) = -z_1 \left[1 + \left(\frac{L}{z_1} \right)^2 \right] = -R_1 \quad c) \end{cases}$$

we have 3 unknowns : z_1, z_2, L and 3 nonlinear equations. the solution of the system a), b), c) is long but straightforward:

$$L = \frac{d(R_1 - d)(R_2 - d)(R_1 + R_2 - d)}{(R_1 + R_2 - 2d)^2}$$

$$\begin{cases} z_1 = \frac{d(R_2 - d)}{R_1 + R_2 - 2d} \\ z_2 = \frac{d(R_1 - d)}{R_1 + R_2 - 2d} \end{cases}$$

- question : we focus our analysis of the spatial profile of the mode, but we did not impose any

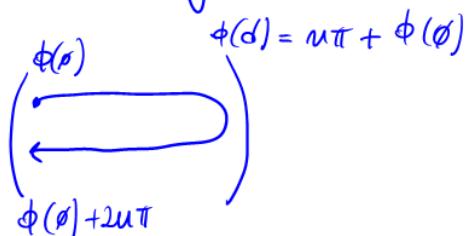
⑥

relation on the phase. What will be the role of the phase in defining the electromagnetic modes in the cavity?

phase of a Gaussian beam:

$$\phi(z) = kz - \operatorname{tg}^{-1}\left(\frac{z}{L}\right)$$

if the beam has to reproduce itself, we need to match the phase as well after one round trip in the cavity:



we therefore have the condition

$$\phi(d) - \phi(0) = kd - \operatorname{tg}^{-1}\left(\frac{d}{L}\right) = q\pi \quad ②$$

let us consider the first resonator:

$$L = \sqrt{dR_2} \sqrt{1 - d/R_2}$$

Equation ② yields a quantization of the allowed wavenumbers of the cavity modes:

$$k_q = \frac{\pi}{d} \left[q + \frac{1}{\pi} \operatorname{tg}^{-1}\left(\frac{\sqrt{d/R_2}}{\sqrt{1-d/R_2}}\right) \right]$$

