# ECE325 Advanced Photonics

## Light propagation in anisotropic crystals
## lesson 1

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# Outline

# Energy and symmetry considerations

In a generic anisotropic medium, the relationship between the polarization vector $\boldsymbol{p}$ and the electric field $\boldsymbol{e}$ reads as $\boldsymbol{p} = \epsilon_0 \underline{\underline{\chi}} \boldsymbol{e}$, with:

$$\underline{\underline{\chi}} = \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix}, \tag{1}$$

being $\underline{\underline{\chi}}$ a rank-2 tensor describing the anisotropic response of the material. The electric displacement then becomes:

$$\boldsymbol{d} = \epsilon_0 \boldsymbol{e} + \boldsymbol{p} = \underline{\underline{\epsilon}} \boldsymbol{e} \tag{2}$$

with:

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}, \tag{3}$$

the anisotropic dielectric tensor.

# Energy and symmetry considerations

- Question: are all the 9 elements of the tensors $\underline{\underline{\chi}}$ and $\underline{\underline{\epsilon}}$ independent?

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To answer to this question we can start from the electromagnetic energy stored in the material:

$$\mathcal{E} = \frac{1}{2}\left(\boldsymbol{e}\cdot\boldsymbol{d} + \boldsymbol{h}\cdot\boldsymbol{b}\right) = \frac{1}{2}\left(e_k\epsilon_{kl}e_l + \mu_0 h_l h_l\right), \tag{4}$$

where we have adopted Einstein summation convention over repeated indices, i.e., $\sum_{j=1}^{3} a_{ij}b_j \equiv a_{ij}b_j$. We then take the time derivative of (4):

$$\dot{\mathcal{E}} = \frac{\epsilon_{kl}}{2}\left(\dot{e}_k e_l + e_k\dot{e}_l\right) + \mu_0\dot{h}_l h_l, \tag{5}$$

where $\dot{A} \equiv \frac{\partial A}{\partial t}$. From the conservation of energy of Maxwell equations:

$$\dot{\mathcal{E}} + \nabla\cdot\left(\boldsymbol{e}\times\boldsymbol{h}\right) = 0, \tag{6}$$

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we have:

$$\frac{\epsilon_{kl}}{2}\left(\dot{e}_k e_l + e_k \dot{e}_l\right) + \mu_0 \dot{h}_l h_l = -\nabla \cdot (\boldsymbol{e} \times \boldsymbol{h}). \tag{7}$$

From the Poynting theorem of Maxwell equations:

$$\nabla \cdot (\boldsymbol{e} \times \boldsymbol{h}) + \boldsymbol{e} \cdot \dot{\boldsymbol{d}} + \boldsymbol{h} \cdot \dot{\boldsymbol{b}} = 0, \tag{8}$$

by substituting into (7), we obtain:

$$\frac{\epsilon_{kl}}{2}\left(\dot{e}_k e_l + e_k \dot{e}_l\right) + \mu_0 \dot{h}_l h_l = \epsilon_{kl} \dot{e}_l e_k + \mu_0 \dot{h}_l h_l, \tag{9}$$

which implies:

**Symmetry condition**

$$\epsilon_{kl} = \epsilon_{lk}$$

The dielectric tensor has only 6 independent elements.

# Energy and symmetry considerations

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- Question: can we simplify more the expression of $\underline{\underline{\epsilon}}$ for our study?

We can start from the expression of the electric energy density

$$\mathcal{W}_e = \frac{1}{2}\epsilon_{kl} e_l e_k = \frac{1}{2}\mathbf{e}^T \cdot \underline{\underline{\epsilon}} \cdot \mathbf{e}, \tag{10}$$

which is a quadratic form. In (10), $^T$ is the transpose operator and $\cdot$ indicates the matrix product. Due to the Hermitian nature of the dielectric tensor $\underline{\underline{\epsilon}} = \underline{\underline{\epsilon}}^\dagger$, we can diagonalize $\underline{\underline{\epsilon}}$ by a similarity transformation $\underline{\underline{\gamma}} = \underline{\underline{t}}^T \underline{\underline{\epsilon t}}$, with $\underline{\underline{t}}$ a new reference system:

$$\mathbf{e} = \underline{\underline{t}}\mathbf{e'}, \tag{11}$$

we have:

$$\mathcal{W}_e = \frac{1}{2}\mathbf{e}'^T \cdot \underline{\underline{t}}^T \underline{\underline{\epsilon t}} \cdot \mathbf{e}' = \frac{1}{2}\mathbf{e}'^T \cdot \underline{\underline{\gamma}} \cdot \mathbf{e}', \tag{12}$$

# Energy and symmetry considerations

with diagonal $\underline{\underline{\gamma}}$:

$$\underline{\underline{\gamma}} = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix}. \tag{13}$$

From (12), $\underline{\underline{\gamma}}$ represents the expression of the new dielectric tensor in the reference system described by $\underline{\underline{t}}$, which is called principal axis reference. In order to construct $\underline{\underline{t}}$ and $\underline{\underline{\gamma}}$, we begin by calculating the eigenvalues and eigenvectors of $\underline{\underline{\epsilon}}$:

$$\underline{\underline{\epsilon}}\boldsymbol{c}_j = \alpha_j \boldsymbol{c}_j, \tag{14}$$

with $\alpha_j$ the eigenvalue and $\boldsymbol{c}_j$ the eigenvector of $\underline{\underline{\epsilon}}$, with $j = 1, 2, 3$. Since the matrix $\underline{\underline{\epsilon}}$ is hermitian, eigenvalues are reals and eigenvectors are orthogonal to each other. The transformation matrix $\underline{\underline{t}}$ is then constructed from:

$$\underline{\underline{t}} = [\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3] \tag{15}$$

# Energy and symmetry considerations

- Exercise: Calculate the expression of $\underline{\underline{\gamma}}$

Answer: if the eigenvectors are normalized, i.e., if $\boldsymbol{c}_i \boldsymbol{c}_j = \delta_{ij}$, we can verify that:

$$\underline{\underline{\gamma}} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}. \tag{16}$$

- Advanced Question: What are the physical implications of the similarity transformation $\underline{\underline{\gamma}}$ on the electromagnetic fields $\boldsymbol{e}$ and $\boldsymbol{h}$ and why is it called similarity transform?

# Outline

# Classification of anisotropic crystals

The analysis described above shows that only a maximum of 3 elements of the dielectric tensor can be really independent. Only the following cases are then possible for an anisotropic crystal:

- $\alpha_1 = \alpha_2 = \alpha_3 \rightarrow$ This represents an isotropic material.
- $\alpha_1 = \alpha_2 \neq \alpha_3 \rightarrow$ This indicates a Uniaxial crystal. The direction corresponding to $\alpha_3$ known as optical axis.
- $\alpha_1 \neq \alpha_2 \neq \alpha_3 \rightarrow$ This is a Biaxial crystal.

# Outline

# Plane waves in anisotropic crystals

Starting from Maxwell equations:

$$\begin{cases} \nabla \times \boldsymbol{h} = \partial_t \boldsymbol{d} \\ \nabla \times \boldsymbol{e} = -\partial_t \boldsymbol{b} \end{cases}, \qquad \partial_t \equiv \frac{\partial}{\partial t}, \tag{17}$$

and looking for plane waves solutions $\boldsymbol{e} = \boldsymbol{E} e^{i\omega t - i\boldsymbol{kr}}$, $\boldsymbol{h} = \boldsymbol{H} e^{i\omega t - i\boldsymbol{kr}}$, we obtain:

$$\begin{cases} \boldsymbol{k} \times \boldsymbol{H} = -\omega \boldsymbol{D} \\ \boldsymbol{k} \times \boldsymbol{E} = \omega \mu_0 \boldsymbol{H} \end{cases}, \tag{18}$$

From these equations we observe that the field $\boldsymbol{D}$ is orthogonal to $\boldsymbol{k}$ and $\boldsymbol{H}$, while the field $\boldsymbol{H}$ is orthogonal to $\boldsymbol{E}$ and $\boldsymbol{k}$.

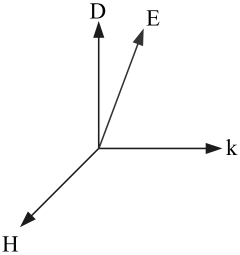
# Plane waves in anisotropic crystals

- Exercise: Prove that $\boldsymbol{D}$, $\boldsymbol{H}$ and $\boldsymbol{k}$ are mutually orthogonal to each other

# Plane waves in anisotropic crystals

- Exercise: Prove that $\boldsymbol{D}$, $\boldsymbol{H}$ and $\boldsymbol{k}$ are mutually orthogonal to each other

We can easily prove it from the divergence equations $\nabla \cdot \boldsymbol{d} = \nabla \cdot \boldsymbol{b} = 0$, which lead to $\boldsymbol{k} \cdot \boldsymbol{D} = \boldsymbol{k} \cdot \boldsymbol{H} = 0$.




- Advanced Question: What are the physical implications of this result?

# Outline

# Reference texts

- A. Yariv, *Photonics: Optical Electronics in Modern Communications* (Oxford University Press, 2006). Chapter 1
- A. Yariv, *Quantum electronics* (Wiley, 1989). Chapter 5