ECE325 Advanced Photonics

Light propagation in anisotropic crystals lesson 2

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Outline

Plane waves in anisotropic crystals

Starting from Maxwell equations:

$$\begin{cases} \nabla \times \boldsymbol{h} = \partial_t \boldsymbol{d} \\ \nabla \times \boldsymbol{e} = -\partial_t \boldsymbol{b} \end{cases}, \qquad \partial_t \equiv \frac{\partial}{\partial t}, \tag{1}$$

We look for plane waves solutions $\boldsymbol{e} = \boldsymbol{E} e^{i\omega t - i\boldsymbol{kr}}$, $\boldsymbol{h} = \boldsymbol{H} e^{i\omega t - i\boldsymbol{kr}}$, with:

$$\boldsymbol{k} = \frac{\omega}{c} n \cdot \hat{s}, \tag{2}$$

being n an equivalent refractive index that characterizes the plane wave propagation and $\hat{s}$ is a unit vector in the direction of $\boldsymbol{k}$. By substituting into (1), we obtain:

$$\begin{cases} \boldsymbol{D} = -\frac{n}{c} \hat{s} \times \boldsymbol{H} \\ \boldsymbol{H} = \frac{n}{\mu_0 c} \hat{s} \times \boldsymbol{E} \end{cases}, \tag{3}$$

Plane waves in anisotropic crystals

By solving for $\boldsymbol{D}$, we have:

$$\boldsymbol{D} = -\frac{n^2}{\mu_0 c^2} \hat{s} \times \hat{s} \times \boldsymbol{E} = -\frac{n^2}{\mu_0 c^2} [\hat{s}(\hat{s} \cdot \boldsymbol{E}) - \boldsymbol{E}], \tag{4}$$

where we have used the double rotor expansion
$A \times B \times C = B(AC) - C(AB)$. Without loss of generality, we can assume the the dielectric tensor is in diagonal form: $D_k = \epsilon_0 \epsilon_k E_k$. With this position, (4) become:

$$E_k = \frac{n^2 s_k (\hat{s} \cdot \boldsymbol{E})}{n^2 - \epsilon_k}, \tag{5}$$

By multiplying by s_k and summing up all terms along x,y, and z, we obtain:

$$s_k E_k = \frac{n^2 s_k^2 (\hat{s} \cdot \boldsymbol{E})}{n^2 - \epsilon_k}, \tag{6}$$

Plane waves in anisotropic crystals

Or equivalently:

$$\hat{s} \cdot \boldsymbol{E} = \frac{n^2 s_k^2 (\hat{s} \cdot \boldsymbol{E})}{n^2 - \epsilon_k}, \tag{7}$$

which implies the following condition

$$\frac{s_x^2}{n^2 - \epsilon_1} + \frac{s_y^2}{n^2 - \epsilon_2} + \frac{s_z^2}{n^2 - \epsilon_3} = \frac{1}{n^2} \tag{8}$$

This is a quadratic equation in n^2, which possess for a given direction defined by $\hat{s}$ two different solutions n_1 and n_2. These indices represent the equivalent refractive indices seen by the plane waves supported by the anisotropic material. Once these indices are found, Eq. (5) furnishes the values of the electric field amplitude of the plane wave.

Plane waves in anisotropic crystals

Summarizing, the plane waves supported by an anisotropic material are calculated from the two equations:

Fesnel equation for the refractive index

$$\frac{s_x^2}{n^2-\epsilon_1}+\frac{s_y^2}{n^2-\epsilon_2}+\frac{s_z^2}{n^2-\epsilon_3}=\frac{1}{n^2} \tag{9}$$

and:

Field amplitude

$$E_k=\frac{n^2 s_k(\hat{s}\cdot\boldsymbol{E})}{n^2-\epsilon_k}, \tag{10}$$

which are to be solved given a spatial direction $\hat{s}$ of the wavevector $\boldsymbol{k}$.

Outline

Generic properties of plane waves solutions of anisotropic media

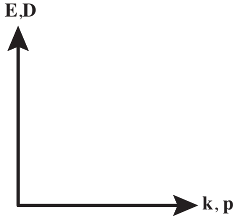
For each frequency ω and direction of wavevector $\hat{s}$, an anisotropic material supports the propagation of two planes waves, each with different refractive index $n_{1,2}$ and phase velocity $v_{1,2} = \frac{c}{n_{1,2}}\hat{s}$. This property makes anisotropic crystals birefringent materials.

birefringence
: a material showing a refractive index depending on the propagation direction and polarization of light.

Generic properties of plane waves solutions of anisotropic media

The plane waves supported by an anisotropic crystal are usually classified according to the corresponding polarization of the electric field. We have the following two classes of possible solutions:

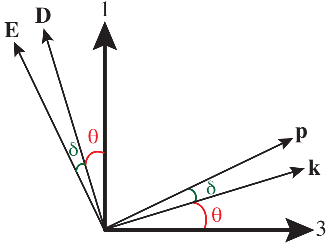
- Ordinary waves, typically denoted as "o". In this case the electric field is parallel to a principal axis p, with $\boldsymbol{D}$ and $\boldsymbol{E}$ parallel: $\boldsymbol{D} = \epsilon_0 \epsilon_p \boldsymbol{E}$



In this case the plane wave behaves as an ordinary plane wave in a material with dielectric constant ϵ_p. The wavevector $\boldsymbol{k}$ is parallel to the Poynting vector $\boldsymbol{p} = \boldsymbol{E} \times \boldsymbol{H}$.

Generic properties of plane waves solutions of anisotropic media

- Extraordinary waves, typically denoted as “e”. In this case the electric field is contained in a plane defined by two principal axis with different value of dielectric constant. Fields ***E*** and ***D*** are no longer parallel, and their vectors form an angle δ known as walk-off angle.



In the example illustrated in the figure, we have $\hat{s} = [\sin\theta, 0, \cos\theta]$ and $\boldsymbol{D} = \epsilon_0[\epsilon_1 E_x, 0, \epsilon_3 E_z]$, which is defined along axis ‘1’ and ‘3’ corresponding to the directions x and z of ϵ_1 and ϵ_3, with $\epsilon_1 \neq \epsilon_3$.

Generic properties of plane waves solutions of anisotropic media

From the index equation (9):

$$\frac{\sin\theta^2}{n^2-\epsilon_1}+\frac{\cos\theta^2}{n^2-\epsilon_3}=\frac{1}{n^2} \tag{11}$$

we can calculate the refractive index n of the extraordinary plane wave:

$$n=\frac{n_1 n_3}{\sqrt{n_3^2\cos\theta^2+n_1^2\sin\theta^2}}, \tag{12}$$

with $n_i \equiv \sqrt{\epsilon_i}$. As a result, the refractive index n seen by the plane wave changes with the direction of $\boldsymbol{k}$ (birefringence) and the direction of propagation of the energy, defined by $\hat{p}$, is not the same of the direction of the wavevector $\boldsymbol{k}$.

Generic properties of plane waves solutions of anisotropic media

Exercise: demonstrate that in the previous example the walk-off angle is:

$$\tan\delta = \frac{n^2}{2}\left(\frac{1}{n_3^2} - \frac{1}{n_1^2}\right) \tag{13}$$

Advanced question: what are the physical implications of a nonzero walk-off angle δ?

Outline

Reference texts

- A. Yariv, *Photonics: Optical Electronics in Modern Communications* (Oxford University Press, 2006). Chapter 1
- A. Yariv, *Quantum electronics* (Wiley, 1989). Chapter 5