# ECE325 Advanced Photonics

## Light propagation in anisotropic crystals
## lesson 3

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# Outline

# Uniaxial crystals

$$\underline{\underline{\epsilon}} = \epsilon_0 \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \tag{1}$$

To calculate the plane wave solutions that can propagate in this material, we begin from the index equation:

$$\frac{s_x^2}{n^2 - n_1^2} + \frac{s_y^2}{n^2 - n_1^2} + \frac{s_z^2}{n^2 - n_3^2} = \frac{1}{n^2} \tag{2}$$

with $n_i = \sqrt{\epsilon_i}$. This equation can be rewritten in the following form:

$$\left(n^2 - n_1^2\right)\left[n_3^2 n_1^2 - n^2\left(n_1^2 s_x^2 + n_1^2 s_y^2 + n_3^2 s_z^2\right)\right] = 0, \tag{3}$$

which admits the following solutions:

# Uniaxial crystals

- $n = n_1$. This is an ordinary wave "o". From the field equation:

$$\left(n^2 - n_k^2\right) E_k = n^2 s_k(\hat{s} \cdot \boldsymbol{E}), \tag{4}$$

with k=1,2,3. we have:

1. For $k = 1$ or $k = 2$, we have $n^2 - n_k^2 = n^2 - n_1^2 = n_1^2 - n_1^2 = 0$, which substituted into (4) leads to

$$\hat{s} \cdot \boldsymbol{E} = 0, \tag{5}$$

   which implies $\boldsymbol{E} \perp \boldsymbol{k}$
2. For $k = 3$, we have conversely:

$$\left(n_1^2 - n_3^2\right) E_z = n^2 s_z(\hat{s} \cdot \boldsymbol{E}) = 0, \tag{6}$$

   which leads to $E_z = 0$.

This is a classical ordinary wave with electric field $\boldsymbol{E}$ lying in the plane defined by the axis 1 and 2, and wavevector $\boldsymbol{k} = \frac{\omega}{c} n_1 \hat{s}$ perpendicular to $\boldsymbol{E}$.

# Uniaxial crystals

- $n = \frac{n_1 n_3}{\sqrt{n_1^2(s_x^2+s_y^2)+n_3 s_z^2}}$. This case yields an extraordinary wave "e". If the wavevector $\boldsymbol{k} = k[\sin\theta, 0, \cos\theta]$ lies in the plane $(x, z)$, we have:

$$n(\theta) = \frac{n_1 n_3}{\sqrt{n_1^2 \sin\theta^2 + n_3^2 \cos\theta^2}}, \tag{7}$$

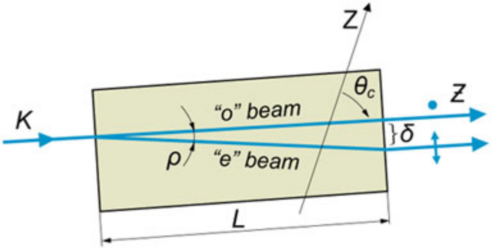
as obtained in the previous lesson. From the field equation (4), we then have the expression of the electric field of the plane wave:

$$E_k = \frac{n^2 s_k \, (\hat{s} \cdot \boldsymbol{E})}{n^2 - n_k^2}. \tag{8}$$

Birefringence is observed in the refractive index $n(\theta)$, which changes with the direction of $\boldsymbol{k}$.

# Uniaxial crystals

In a Uniaxial crystal, for a given frequency and wavevector $\boldsymbol{k}$, we therefore have 2 plane wave solutions: one represented by an ordinary wave, and another represented by an extraordinary wave.




Advanced question: in the case of localized beams given by the superposition of collimated plane waves, such as Gaussian beams, what are the physical implications of this result?

# Outline

# The index ellipsoid

This is powerful graphical method that is extensively used. It is completely equivalent to the previous formulation. We begin by expressing the iso-energy surfaces, obtained from the dielectric energy density:

$$\mathcal{W}_e = \frac{1}{2}\boldsymbol{e}^T \cdot \underline{\underline{\epsilon}} \cdot \boldsymbol{e} = \frac{e_x^2 \epsilon_1}{2} + \frac{e_y^2 \epsilon_2}{2} + \frac{e_z^2 \epsilon_3}{2}. \tag{9}$$

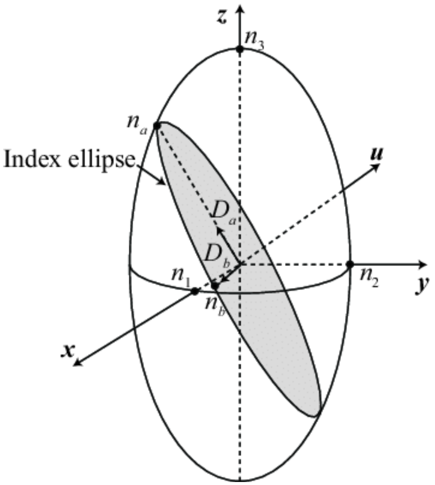
By defining the following 'position' vector $\boldsymbol{r} \equiv [x, y, z] = \frac{\boldsymbol{d}}{\sqrt{2\mathcal{W}_e}}$, with $\boldsymbol{d} = \underline{\underline{\epsilon}} \cdot \boldsymbol{e}$ the dielectric displacement, we obtain the index ellipsoid equation:

$$\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} = 1. \tag{10}$$

This equation, which describes an ellipsoid in the space defined by the position vector $\boldsymbol{r}$, can be used to find the index $n$ and the direction of polarization of the displacement $\boldsymbol{d}$ of the plane waves supported by the anisotropic material.

# The index ellipsoid

1. Find the intersection ellipse between a plane through the origin that is normal to the direction of wavevector $\hat{s}$ and the ellipsoid (gray area in the figure).
2. The two axes of the intersection ellipse have semi-lengths $n_1$ and $n_2$, corresponding to the index of the plane waves supported by the material.
3. The two semi-axis of the intersection ellipse are parallel to the direction of **d** of the plane waves of the anisotropic crystal.



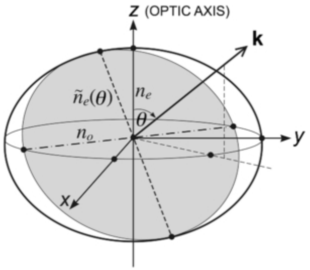
# Outline

# Uniaxial crystals: again

Exercise: study Uniaxial crystals with the index ellipsoid method. The index ellipsoid equation reads as follows:

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1, \tag{11}$$

with $n_o$ and $n_e$ representing the ordinary and the extraordinary index, respectively.

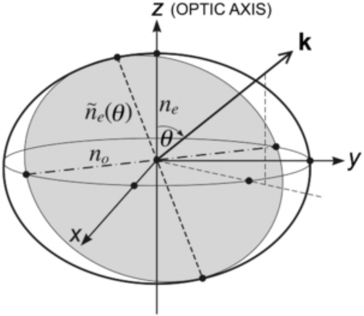


# Uniaxial crystals: again

For any direction of $\boldsymbol{k}$, one semi-axis of the intersection ellipse has always length $n = n_o$, corresponding to an ordinary wave, while the other is an extraordinary wave with $n = \tilde{n}_e(\theta)$:

$$\tilde{n}_e(\theta) = \frac{n_o n_e}{\sqrt{n_e^2 \sin\theta^2 + n_o^2 \cos\theta^2}} \qquad (12)$$

The refractive index associated to the extraordinary wave takes any value between $n_e$ and $n_o$, depending on the direction of $\boldsymbol{k}$.



# Uniaxial crystals: again

Exercise: consider a Uniaxial crystal with length $L$ along $z$ and optic axis in the plane $(x, y)$. A plane wave with $\boldsymbol{k}$ parallel to $\hat{z}$ impinges on the crystal. Calculate the emerging Electric field at the output of the crystal.

# Outline

# Reference texts


- A. Yariv, *Photonics: Optical Electronics in Modern Communications* (Oxford University Press, 2006). Chapter 1
- A. Yariv, *Quantum electronics* (Wiley, 1989). Chapter 5