ECE325 Advanced Photonics

Light propagation in anisotropic crystals
lesson 4

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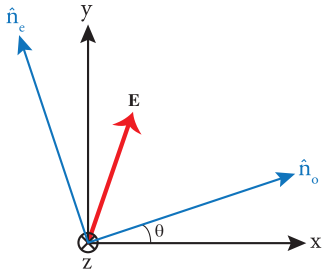
Outline

Propagation of light in Uniaxial crystals

Exercise: consider a Uniaxial crystal with length L along z and optic axis in the plane (x, y). A plane wave with $\boldsymbol{k}$ parallel to $\hat{z}$ impinges on the crystal. Calculate the emerging Electric field at the output of the crystal.



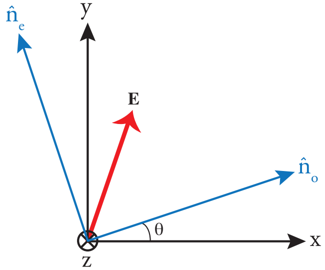
$\hat{n}_o$ and $\hat{n}_e$ are unit vectors along the principal ordinary and extraordinary axes, respectively, while $\boldsymbol{E} = [E_x, E_y, 0]$ is electric input field.

Propagation of light in Uniaxial crystals

For $\boldsymbol{k}$ parallel to z, the plane waves propagating in the uniaxial crystals have refractive index n_e and n_o, with polarization vectors parallel to $\hat{n}_e$ and $\hat{n}_o$, respectively. To solve for the propagation of the electric field, we decompose the field along $\hat{n}_e$ and $\hat{n}_o$:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} E_o \\ E_e \end{bmatrix} \quad (1)$$

Each component E_e and E_o represents a plane wave solution inside the anisotropic crystal propagating with wavevector $\boldsymbol{k}_e = \frac{\omega}{c} n_e \hat{z}$ and $\boldsymbol{k}_o = \frac{\omega}{c} n_o \hat{z}$, respectively.



Propagation of light in Uniaxial crystals

After a distance $z = L$ in the uniaxial crystal, plane waves propagate as follows:

$$\begin{aligned} E_o(L) &= E_o(0)e^{-ik_{oz}L} = E_o e^{-i\frac{\omega}{c}n_o L} \\ E_e(L) &= E_e(0)e^{-ik_{ez}L} = E_e e^{-i\frac{\omega}{c}n_e L} \end{aligned} \qquad (2)$$

In matrix form, the last expression reads:

$$\begin{bmatrix} E_o(L) \\ E_e(L) \end{bmatrix} = e^{-i\frac{\omega}{c}L\cdot\underline{\underline{n}}} \begin{bmatrix} E_o(0) \\ E_e(0) \end{bmatrix}, \qquad \underline{\underline{n}} = \begin{bmatrix} n_o & 0 \\ 0 & n_e \end{bmatrix} \qquad (3)$$

Due to birefringence, the two plane waves, originally having the same phase, acquire a phase delay $\Delta\phi = (n_o - n_e)\frac{\omega L}{c}$ at $z = L$.

Propagation of light in Uniaxial crystals

It is convenient to write the propagation of the field as a function of the phase delay $\Delta\phi$:

$$\begin{bmatrix} E_o(L) \\ E_e(L) \end{bmatrix} = e^{-i\frac{1}{2}(n_e+n_o)\frac{\omega}{c}L} \begin{bmatrix} e^{-i\frac{\Delta\phi}{2}} & 0 \\ 0 & e^{i\frac{\Delta\phi}{2}} \end{bmatrix} \begin{bmatrix} E_o(0) \\ E_e(0) \end{bmatrix} = \underline{\underline{W}} \cdot \begin{bmatrix} E_o(0) \\ E_e(0) \end{bmatrix} \qquad (4)$$

At the output of the crystal, in order to get back to the original coordinates, we apply a rotation:

$$\begin{bmatrix} E_x(L) \\ E_y(L) \end{bmatrix} = \underline{\underline{R}} \cdot \underline{\underline{W}} \cdot \underline{\underline{R}}^{\dagger} \cdot \begin{bmatrix} E_x(0) \\ E_y(0) \end{bmatrix}, \qquad (5)$$

with:

$$\underline{\underline{R}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad (6)$$

defining a rotation matrix in the (x, y) plane.

Propagation of light in Uniaxial crystals

The total input-output transfer matrix of the structure is then:

$$\underline{\underline{R}} \cdot \underline{\underline{W}} \cdot \underline{\underline{R}}^{\dagger} = \begin{bmatrix} e^{-i\frac{\Delta\phi}{2}} \cos\theta^2 + e^{i\frac{\Delta\phi}{2}} \sin\theta^2 & -i \sin\frac{\Delta\phi}{2} \sin 2\theta \\ -i \sin\frac{\Delta\phi}{2} \sin 2\theta & e^{-i\frac{\Delta\phi}{2}} \sin\theta^2 + e^{i\frac{\Delta\phi}{2}} \cos\theta^2 \end{bmatrix} \quad (7)$$

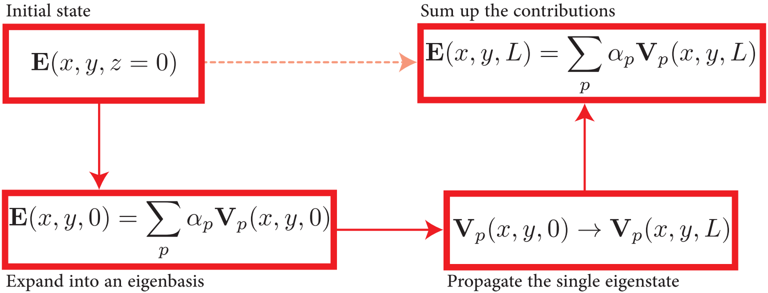
The approach used above to the model the propagation of electric field in anisotropic crystals is known as Jones Calculus, while the 2×2 transfer matrices that we used in the previous equations are known as Jones Matrices.

Exercise: verify that the transfer matrix is unitary, i.e., $\underline{\underline{T}} \cdot \underline{\underline{T}}^{\dagger} = \underline{\underline{1}}$, with $\underline{\underline{1}}$ being the identity matrix and $\underline{\underline{T}} = \underline{\underline{R}} \cdot \underline{\underline{W}} \cdot \underline{\underline{R}}^{\dagger}$.

Advanced question: what is the physical interpretation of this result?

Propagation of light in Uniaxial crystals

From a more general perspective, the approach we used to solve the propagation problem is based on eigenvalue decomposition:



and exploits the linearity of Maxwell equations. For nonlinear materials, this approach cannot be used and other strategies are employed based on more complex transforms.

Outline

Half-wave retarder plate

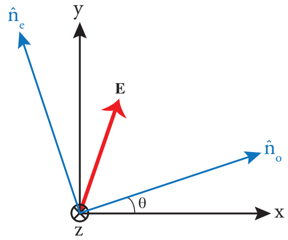
It is composed by a uniaxial crystal in the configuration studied in the previous exercise, with:

$$\Delta\phi = (n_e - n_o)\frac{\omega}{c}L = \pi, \quad (8)$$

The length of the crystal is then $L = \frac{\lambda}{2\Delta n}$, with $\Delta n = n_e - n_o$. In this system, when $\theta = \frac{\pi}{4}$, the transfer matrix $\underline{\underline{T}}$ reads:

$$\underline{\underline{T}} = -i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -i\sigma_1, \quad (9)$$

being σ_1 the Pauli matrix.

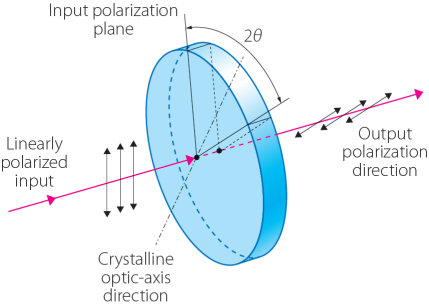


Half-wave retarder plate

An $x-$polarized electric field at the beginning $\boldsymbol{E} = [1, 0]$, transforms into:

$$\boldsymbol{E}(L) = -i\sigma_1 \boldsymbol{E} = -i \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{10}$$

which represents a $y-$polarized field at the output.



Performance of half (λ/2) waveplate

Half-wave retarder plate

Exercise: calculate the evolution of a circular polarized electric field $\boldsymbol{E} = \frac{1}{\sqrt{2}}[1, i]$

Advanced question: design a power controller by using an half-wave retarder.

Half-wave retarder plate

Exercise: calculate the evolution of a circular polarized electric field $\boldsymbol{E} = \frac{1}{\sqrt{2}}[1, i]$

Advanced question: design a power controller by using an half-wave retarder. We can use a half-wave retarder plate followed by a polarizer. We can verify that the transfer matrix of a polarizer is:

$$\underline{\underline{P}}_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \underline{\underline{P}}_y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{11}$$

for an $x-$polarizer and $y-$ polarizer, respectively. If we orient the half-wave retarder plate at a generic θ, the transfer matrix becomes:

$$\underline{\underline{T}} = -i \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \tag{12}$$

Half-wave retarder plate

Consider a generic field at the input linearly polarized, in this example along x, $\boldsymbol{E} = [E_x, 0]$. At the output of the system crystal + polarizer, we have:

$$\begin{bmatrix} E_x(L) \\ E_y(L) \end{bmatrix} = \underline{\underline{P}}_x \cdot \underline{\underline{T}} \cdot \begin{bmatrix} E_x \\ 0 \end{bmatrix} = \begin{bmatrix} E_x \cos 2\theta \\ 0 \end{bmatrix}, \tag{13}$$

as a result, the output intensity $I = |\boldsymbol{E}|^2 = |E_x|^2 \cos 2\theta^2$ cab be modulated between 0 and $|E_x|^2$ by simply acting on the rotation angle of the crystal. When $\theta = \frac{\pi}{4}$, the output intensity is 0, as expected. In this condition, in fact, the half-wave retarder rotates the input polarization and the polarizer prevents the beam to pass through the system.

Quarter-wave retarder plate

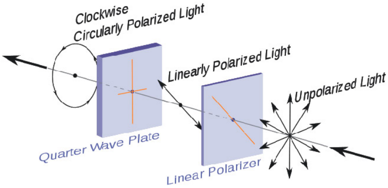
It is composed by a uniaxial crystal with:

$$\Delta\phi = (n_e - n_o)\frac{\omega}{c}L = \frac{\pi}{2}, \qquad (14)$$

The length of the crystal is then $L = \frac{\lambda}{4\Delta n}$, with $\Delta n = n_e - n_o$. In this system, when $\theta = \frac{\pi}{4}$, the transfer matrix $\underline{\underline{T}}$ reads:

$$\underline{\underline{T}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \qquad (15)$$

This transforms linearly polarized beams into circularly polarized light and vice-versa.



Retarder plates

Advanced question: A student is investigating the propagation of light in a system composed by two polarizers in cross configuration. Due to this configuration, no light is expected to emerge from the system. However, the student observes that if he puts a half-retarder plate in between the two polarizers, some light emerges from the system. Explain the phenomenon.

Outline

Reference texts

- A. Yariv, *Photonics: Optical Electronics in Modern Communications* (Oxford University Press, 2006). Chapter 1
- A. Yariv, *Quantum electronics* (Wiley, 1989). Chapter 5