ECE325 Advanced Photonics

Waveguide theory
lesson 11

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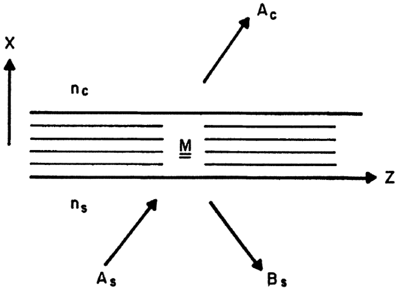
Outline

1. Study of planar waveguides via multilayer theory
   - Transfer matrix theory
   - TE Modes
   - TM Modes

2. Reference texts

1D Multilayer theory

We study general planar waveguides by a transfer matrix approach, representing the refractive index $n(x)$ as a stratified material composed of a succession of discrete layers, each with thickness h_i and constant refractive index n_i. This theory can also describe light reflection and transmission in single and multi-layer structure assembled by layers of different refractive index, such as anti-reflection coatings and dielectric mirrors.



TE case

We begin by writing Maxwell's equations for a TE wave with nonzero components E_y, H_x, H_z in a generic layer with refractive index n. Assuming propagation along z and symmetry along y, we have $\partial_z \rightarrow -i\beta$, $\partial_y \rightarrow 0$, $\partial_t \rightarrow i\omega$, and:

$$\begin{cases} -\partial_x H_z - i\beta H_x = i\omega\epsilon_0 n^2 E_y, \\ \partial_x E_y = -i\omega\mu_0 H_z, \\ i\beta E_y = -i\omega\mu_0 H_x \end{cases} \tag{1}$$

with $\beta = k_0 n_{eff}$. By solving the last equation for $H_x = -\frac{\beta}{\omega\mu_0}E_y$, we can express the system as a function of E_y and H_z:

$$\begin{cases} H_z = \frac{i}{\omega\mu_0}\partial_x E_y, \\ i\omega\mu_0\partial_x H_z = k_0^2(n^2 - n_{eff}^2)E_y \end{cases} \tag{2}$$

TE case

We can simplify (2) by introducing the adimensional quantities:

$$U = \frac{E_y}{E_0}, \quad V = i\frac{\omega\mu_0}{k_0 E_0}, \quad x \rightarrow \frac{x}{k_0}, \tag{3}$$

with arbitrary constant electric field E_0, into:

Single layer final system

$$\begin{cases} \partial_x U = -V, \\ \partial_x V = (n^2 - n_{eff}^2) \cdot U \end{cases} \tag{4}$$

whose solution reads as follows:

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \begin{bmatrix} \cos(\gamma x) & \frac{\sin(\gamma x)}{\gamma} \\ -\gamma \sin(\gamma x) & \cos(\gamma x) \end{bmatrix} \cdot \begin{bmatrix} U(x) \\ V(x) \end{bmatrix} = \underline{\underline{M}}(x) \cdot \begin{bmatrix} U(x) \\ V(x) \end{bmatrix} \tag{5}$$

with $\gamma = \sqrt{n^2 - n_{eff}^2}$, $U_0 = U(0)$ and $V_0 = V(0)$.

TE case

In the m-th layer, we have:

$$\begin{bmatrix} U_{m-1} \\ V_{m-1} \end{bmatrix} = \begin{bmatrix} \cos(\gamma_m h_m) & \frac{\sin(\gamma_m h_m)}{\gamma_m} \\ -\gamma_m \sin(\gamma_m h_m) & \cos(\gamma_m h_m) \end{bmatrix} \cdot \begin{bmatrix} U_m \\ V_m \end{bmatrix} = \underline{\underline{M}}_m \cdot \begin{bmatrix} U_m \\ V_m \end{bmatrix} \qquad (6)$$

with $\gamma_m = \sqrt{n_m^2 - n_{eff}^2}$. The input-output field in a multilayer composed by $m = 1, ..., M$ planar layers is then expressed as:

Multilayer input-ouput field

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \prod_{m=1}^{M} \underline{\underline{M}}_m \begin{bmatrix} U_M \\ V_M \end{bmatrix} = \underline{\underline{M}} \begin{bmatrix} U_M \\ V_M \end{bmatrix} \qquad (7)$$

with $\underline{\underline{M}}$ the transfer matrix of the multilayer.

Waveguide modes

Equation (7) allows to express the electromagnetic field in any point of the space once the effective index n_{eff} is computed. The effective index is computed by specifying boundary conditions at $\pm\infty$ on the radiation field outside the multilayer (radiating boundary conditions). In the semi-infinite regions along x outside the multilayer, for $-\infty < x \leq 0$ and $h \leq h < \infty$ with $h = \sum_m h_m$ the total thickness of the multilayer, the electromgnetic field $U(x)$ and $V(x)$ are expressed by Eqs. (5), with $\gamma = \sqrt{n^2 - n_{eff}^2}$ and $n = n_s$ for $x \leq 0$, $n = n_c$ for $x \geq h$. In the case of guided modes, the field outside the guiding structure (multilayer) should be evanescent, and this implies that γ in $x \leq 0$ and $x \geq h$ is purely imaginary.

Dispersion relation

We can therefore express the field in these regions more conveniently as follows:

$$\begin{cases} U(x) = Ae^{\kappa x} + Be^{-\kappa x}, \\ V(x) = \kappa \left(-Ae^{\kappa x} + Be^{-\kappa x}\right), \end{cases} \tag{8}$$

with $\kappa^2 = -\gamma^2$. For guided modes, the field should decay away from the multilayer, and this implies:

$$U_0 = A, \quad V_0 = -\kappa_s A, \quad U_M = B, \quad V_M = \kappa_c B \tag{9}$$

with $\kappa_s = \sqrt{n_{eff}^2 - n_s^2}$ and $\kappa_c = \sqrt{n_{eff}^2 - n_c^2}$. Inserting these boundary conditions in the input-output relationship (7):

$$\begin{cases} A = (M_{11} + \kappa_c M_{12})B, \\ -\kappa_s A = (M_{21} + \kappa_c M_{22})B \end{cases} \tag{10}$$

with $M_{ij} = (\underline{\underline{M}})_{ij}$

Dispersion relation

By diving numerator and denominator, we obtain the desired dispersion relation for the multilayer slab waveguide:

TE dispersion relation

$$\kappa_s M_{11} + \kappa_c M_{22} + M_{21} + \kappa_s \kappa_c M_{12} = 0$$

expressed in terms of decaying constants κ_c, κ_s and the elements of the transfer matrix for the stack. The solution of this equation furnishes the values of the effective index n_{eff} of all guided modes. Once the effective index is known, Eqs. (5)-(7), (8)-(9) express the corresponding electromagnetic fields in all the region of the space.

Exercise: Compute the dispersion relation of a slab waveguide and verify that it furnishes the same expression obtained via ray optics for TE modes

TM modes

In the case of a TM field, with nonzero components H_y, E_x and E_z, the theory proceeds as in the TE case. By solving the relevant system of equation, we obtain the same solution of the TE case with the substitution $\gamma \rightarrow \frac{\gamma}{n^2}$. This implies that the dispersion relation for TM modes is:

TM dispersion relation

$$\frac{\kappa_s}{n_s^2} M_{11} + \frac{\kappa_c}{n_c^2} M_{22} + M_{21} + \frac{\kappa_s \kappa_c}{n_s^2 n_c^2} M_{12} = 0$$

The complete theory of the multilayer, including the calculation of reflection and transmission coefficients is found on the Tamir book in the references. Exercise: writes a program that, given at the input a multilayer structures with a sequence of M n_i and h_i stack elements, compute the effective indices n_{eff} and the modal profiles of all TE and TM guided modes.

Outline

Reference texts

- T. Tamir, *Guided-wave optoelectronics* (Springer, 1988). Sec. 2.3.3.