ECE325 Advanced Photonics

Waveguide theory
lesson 8

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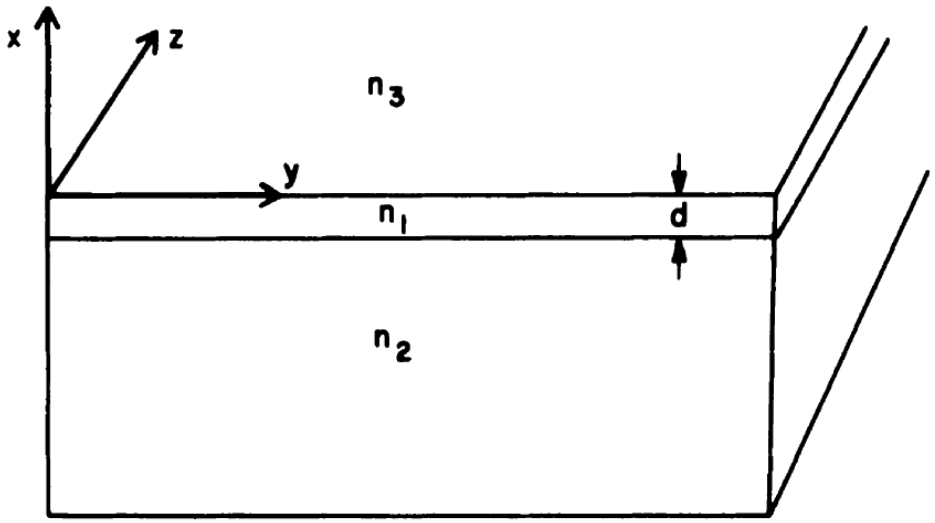
Outline

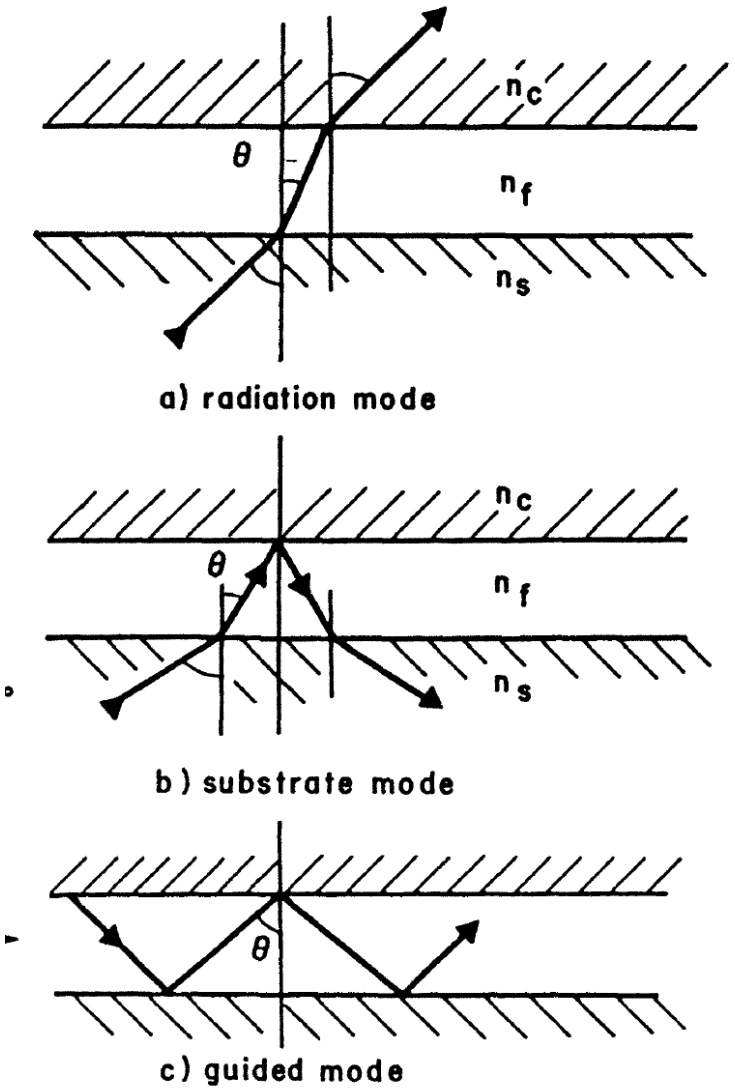
Optical waveguides

An optical waveguide is a photonic structures able to guide light energy inside its structure. Optical waveguides are characterized by a geometry that is symmetric along a direction in space, which identifies the propagation direction of energy. A slab waveguide is the simplest type of waveguide, symmetric along two directions (y and z) and with refractive index changing along x. The guided region has index n_1, thickness d and is surrounded by two semi-infinite regions of constant refractive indices n_2 and n_3. We assume that $n_1 > n_2 \geq n_3$.

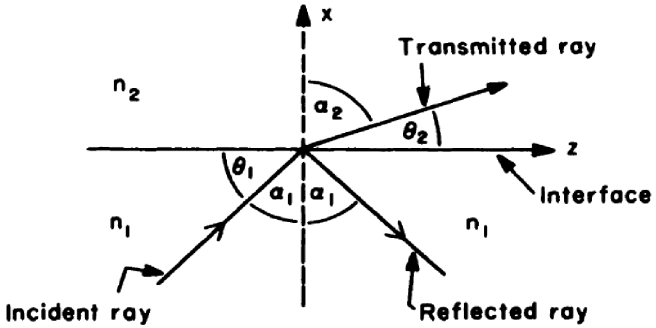


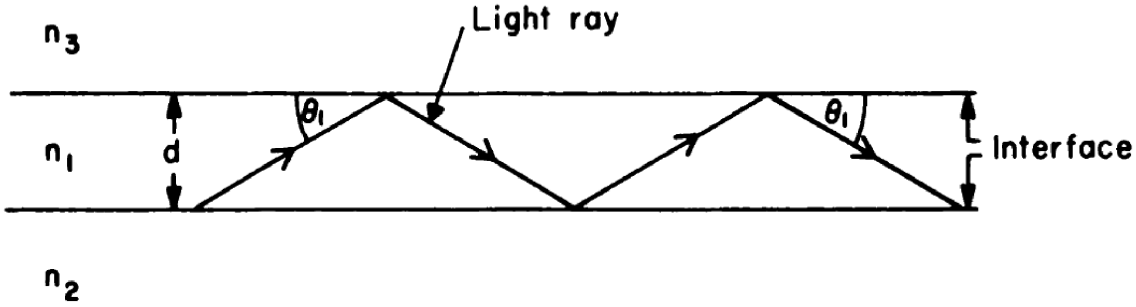
Waveguides modes

Waveguides support the propagation of radiation energy via waveguides modes. A guided mode is a particular wave characterized by a constant intensity profile, and an electromagnetic field varying as $\sim e^{i(\beta y - \omega t)}$, with y the propagation direction of the field, β the propagation constant of the mode and ω light's frequency. The simplest description of modes in a slab waveguide can formulated via ray optics. While ray optics is in general an approximation of waveguide optics, it provides an exact description of modes for dielectric slab waveguides.



Ray optics description of slab waveguide modes





A generic ray is trapped entirely in the guided region of refractive index n_1 if the angle θ_1 is smaller than the critical angle for total internal reflection:

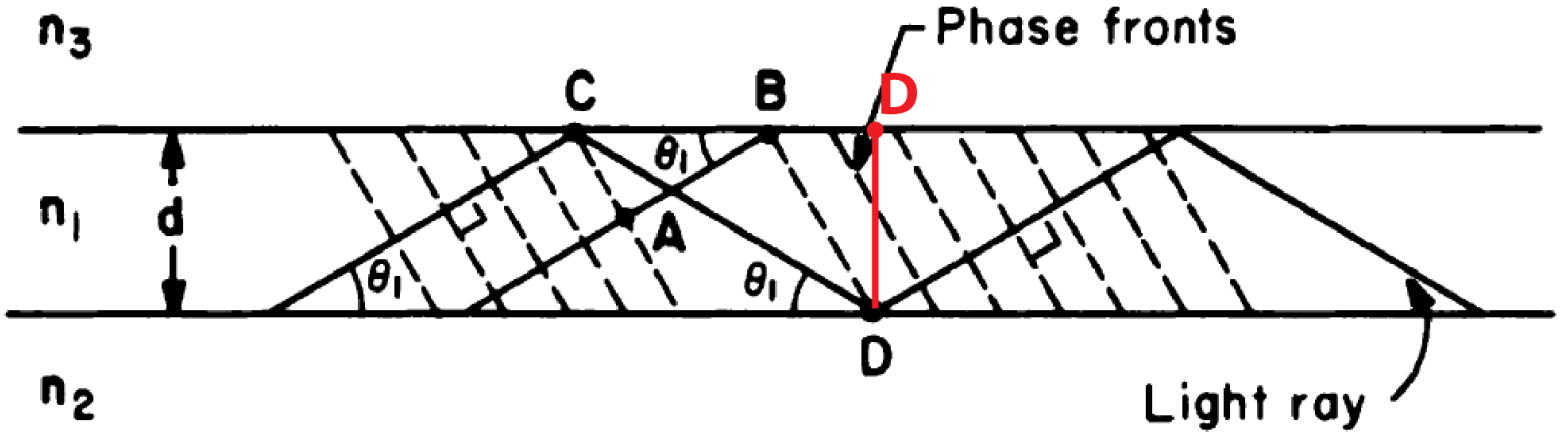
$$n_1 \sin\alpha_1 \geq n_2, \quad \text{and} \quad n_1 \sin\alpha_1 \geq n_3, \tag{1}$$

where we have introduced the complementary angle $\alpha_1 = \frac{\pi}{2} - \theta_1$, measured from the normal of slab surface. From a wave perspective, each ray represents a plane wave with wavevector $\boldsymbol{k} = k\hat{s}$, with $\hat{s}$ the direction of the ray. This implies that the plane wave associated to the ray in figure evolves along y as $e^{ik_y y} = e^{i(k \cdot n_1 \sin\alpha_1 \cdot y)}$. The propagation constant β of the ray is then $\beta = k \cdot n_1 \sin\alpha_1 = k \cdot n_{eff}$. The quantity $n_1 \sin\alpha_1 = n_{eff}$ acts like an effective index seen by the plane wave when propagating inside the material.

Ray optics description of slab waveguide modes

The condition imposed by Eq. (1) is a necessary condition for a ray to be a guided mode, but is not itself sufficient. This because it does not guarantee that the electromagnetic field associated to the ray evolves at every y as $\sim e^{i(\beta y - \omega t)}$, with β constant.

This condition is illustrated in the figure below. The points A and C belong to the same front of the plane wave, as well as the points B and D. Since all points of a phase front of a plane wave must be in phase, we must require that the optical path length of the ray AB differs from the ray CD by an integer multiple of 2π.



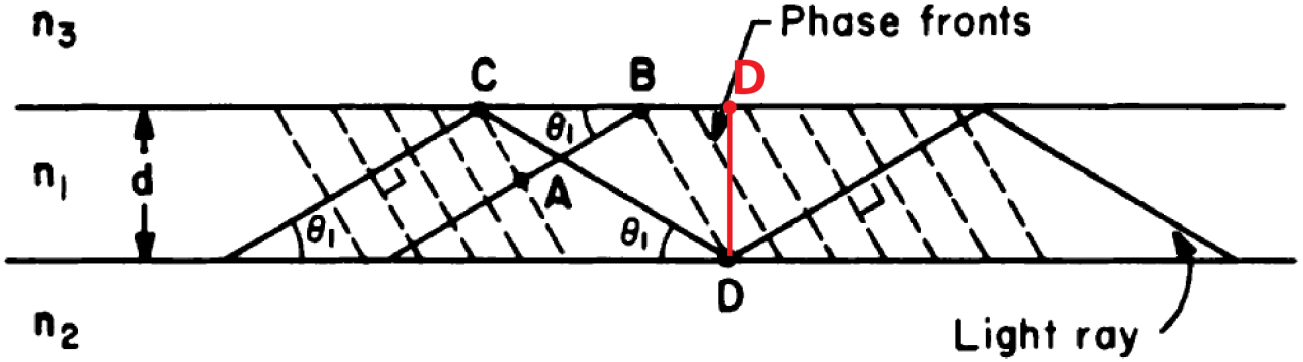
Ray optics description of slab waveguide modes

This condition is written as follows:

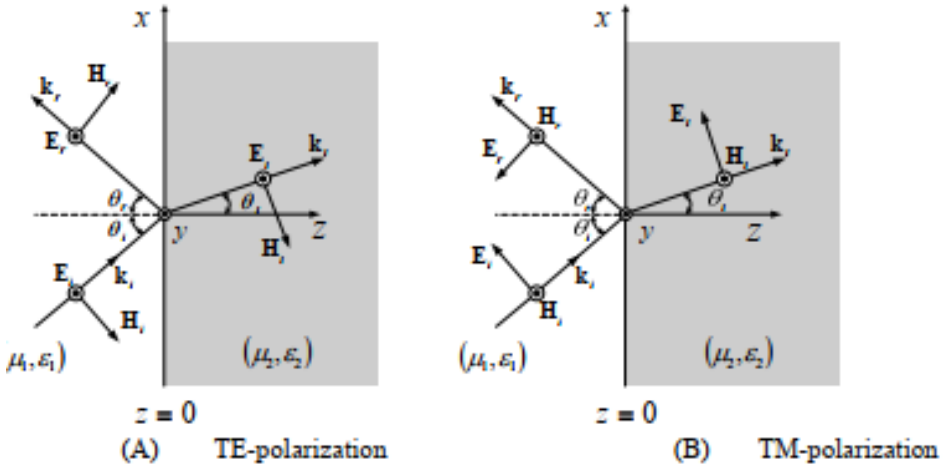
$$\Delta\phi_{CD-AB} + \phi_C + \phi_D = 2m\pi, \quad m = 0, \pm 1, ... \tag{2}$$

with:
1) $\Delta\phi_{CD-AB} = kn_1(\ell_{CD} - \ell_{AB})$ the phase difference between the paths AB and CD, with $\ell_{CD} = \frac{d}{\sin\theta_1}$ the length of the ray CD, $\ell_{AB} = \ell_{CB}\sin\alpha_1$ the length of AB with $\ell_{CB} = \ell_{CD} - \ell_{BD} = \frac{d}{\tan\theta_1} - d\tan\theta_1$. This implies:

$$\Delta\phi_{CD-AB} = 2kn_1 d\cos\alpha_1 \tag{3}$$



Ray optics description of slab waveguide modes



2) ϕ_C and ϕ_D the phase shifts (Fresnel formulas) at points C and D. In TIR condition we have $\sin\alpha_t = \frac{\sin\alpha_i}{n}$, $\cos\alpha_t = i\sqrt{\frac{\sin^2\alpha_t}{n^2} - 1}$, $n = n_2/n_1$, and:

$$\begin{cases} R_{TE} = \frac{E_r}{E_t} = \frac{q}{q^\dagger} = e^{i2\delta_{TE}}, \quad q = \cos\alpha_i - i\sqrt{\sin^2\alpha_i - n^2}, \\ R_{TM} = \frac{E_r}{E_t} = \frac{z}{z^\dagger} = e^{i2\delta_{TM}}, \quad z = n^2\cos\alpha_i - i\sqrt{\sin^2\alpha_i - n^2}, \end{cases} \tag{4}$$

leading to:

$$\delta_{TE} = -\tan^{-1}\left(\frac{\sqrt{\sin^2\alpha_i - n^2}}{\cos\alpha_i}\right), \delta_{TM} = -\tan^{-1}\left(\frac{\sqrt{\sin^2\alpha_i - n^2}}{n^2\cos\alpha_i}\right) \tag{5}$$

TE Modes dispersion relation, symmetric waveguide

We will begin our study with symmetric waveguide $n_2 = n_3 \equiv n$ and for TE modes. The discussion of TM and asymmetric case does not show any qualitative difference.
By substituting Eqs. (3)-(5) for TE polarization into (2), we obtain:

$$2kn_1 d\xi = m\pi + 2\tan^{-1}\sqrt{\frac{n_1^2 - n^2 - n_1^2\xi^2}{n_1^2\xi^2}}, \quad \xi = \cos\alpha_1, \tag{6}$$

which can be written in the following universal form:

$$V\sqrt{1-b} = m\pi + 2\tan^{-1}\sqrt{\frac{b}{1-b}}, \quad \begin{cases} V = 2kd\sqrt{n_1^2 - n^2}, \\ b = 1 - \frac{n_1^2\xi^2}{n_1^2 - n^2}, \end{cases} \tag{7}$$

with $V = 2kd\sqrt{n_1^2 - n^2}$ the normalized frequency and $b = \frac{n_{eff}^2 - n^2}{n_1^2 - n^2}$ the normalized equivalent refractive index. Both V and b are adimensional.

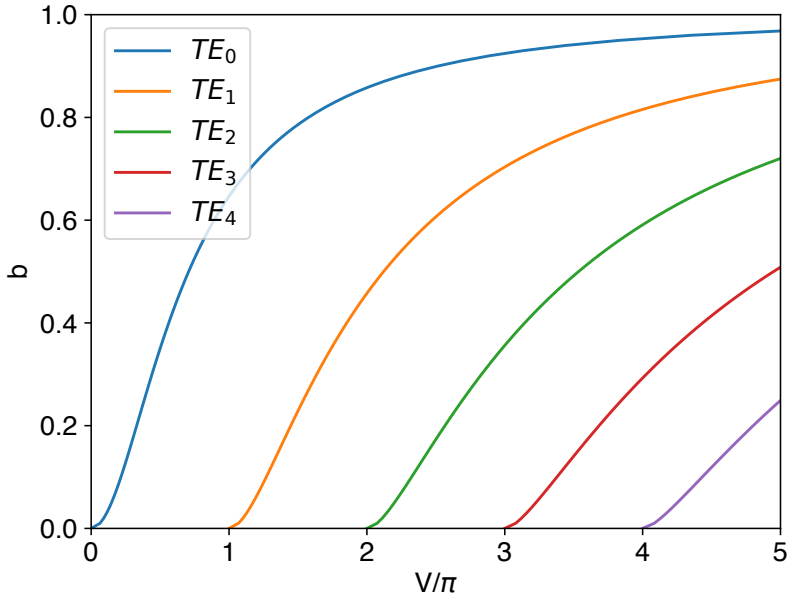
TE Modes dispersion relation, symmetric waveguide

Equation (7) is the dispersion relation of guided TE modes in a symmetric waveguide. It is a dispersion relation as it relates the frequency of the mode $V(\omega = ck)$ with the mode propagation constant $\beta = k \cdot n_{eff}$.

$$V\sqrt{1-b} = m\pi + 2\tan^{-1}\sqrt{\frac{b}{1-b}}, \tag{8}$$

The figure on the right shows the plot V(b) for the first $m = 0, 1, 2, 3, 4$ TE modes of the slab waveguide. The index $0 \leq b < 1$, which implies $n \leq n_{eff} \leq n_1$. The number N of modes supported by the waveguide is given by:

$$N = \text{int}\left(\frac{V}{\pi}\right) \tag{9}$$



Dispersion relation of TM Modes

By repeating the same analysis for TE modes, and by using the Fresnel formulas for TM phase shifts in Eq. (2), we obtain:

$$V\sqrt{1-b} = m\pi + 2\tan^{-1}\left(\frac{n_1^2}{n^2}\sqrt{\frac{b}{1-b}}\right), \tag{10}$$

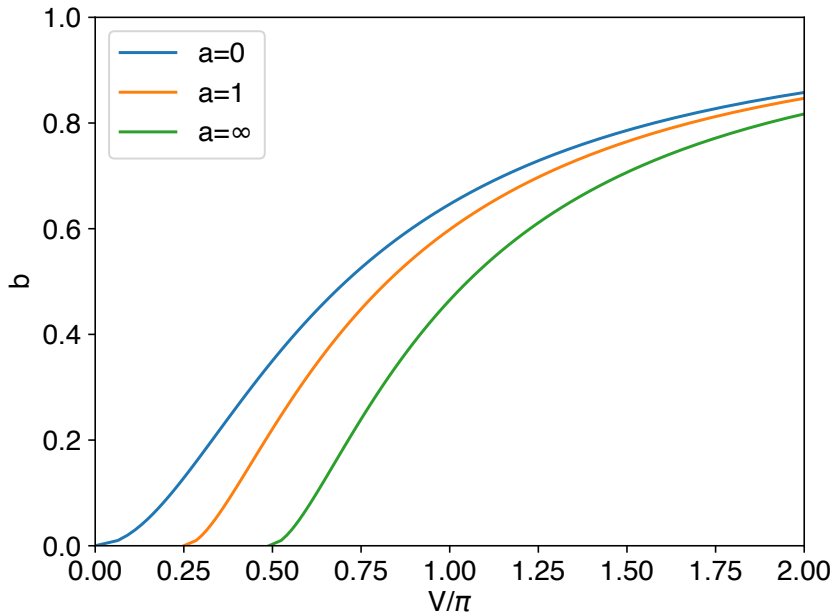
At variance with the TE case, this expression is no longer universal and contains the material properties in the ratio n_1/n. However, in many cases $n_1/n \ll 1$ and the universal TE curve can be used to a good approximation for studying TM modes. No qualitative difference exist with the TE case.

Asymmetric slab waveguides

The dispersion relation from Eq. (2) reads:

$$V\sqrt{1-b} = m\pi + \tan^{-1}\sqrt{\frac{b}{1-b}} + \tan^{-1}\sqrt{\frac{a+b}{1-b}}, \qquad (11)$$

with $a = \frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}$ the asymmetry parameter varying between 0 (symmetric case) and ∞.



Differently from the symmetric case, a cutoff frequency appears in the dispersion for each mode to exist:

$$V(0) = m\pi + \tan^{-1}\sqrt{a} \qquad (12)$$

In the largest asymmetric case $a \to \infty$, $V(0) = \pi(m + \frac{1}{2})$, which implies a cutoff frequency for all modes, including the TE_0 at $m = 0$.

Exercises & Questions

1. Write a program that, for a slab waveguide with given wavelength, thickness d, n_1, n_2 and n_3 calculates the effective indices n_{eff} of all the guided modes propagating in the structure
2. Calculate the numerical aperture of a microscope objective that could excite the propagation of guided modes into a symmetric slab waveguide of given indices n_1 and n_2, thickness d, and at a specific frequency ω.
3. Given the effective index n_{eff} of a guided mode, how to calculate the corresponding electromagnetic field?

Outline

Reference texts

- T. Tamir, *Guided-wave optoelectronics* (Springer, 1988). Chapter 2.
- Dietrich Marcuse, *Theory of Dielectric Optical Waveguides (Second Edition)* (Academic Press, 1991). Chapter 1.