ECE325 Advanced Photonics

Coupled Mode Theory in Space
lesson 12

Andrea Fratalocchi

www.primalight.org

April 17, 2022

Outline

1. General Formulation
2. Data-driven expansion and dimensionality reduction
3. The CMT equations
4. General properties of CMT equations
5. General considerations
6. Reference texts

General idea and starting equations

The coupled mode theory (CMT) in space is an **exact** formalism for studying the propagation of light in waveguides geometries characterized by perturbations of refractive index of arbitrary form and strength, including nonlinear optical effects. Although in modern texts the derivation of the CMT requires several approximations, those are totally unnecessary.

We begin by considering a general 2D waveguide with propagation on the z axis, and decompose the electromagnetic field into a transverse t and longitudinal z component:

$$\begin{cases} \boldsymbol{E} = \boldsymbol{E}_t + \boldsymbol{E}_z, \\ \boldsymbol{H} = \boldsymbol{H}_t + \boldsymbol{H}_z, \end{cases} \tag{1}$$

We then expand the dielectric permittivity $\epsilon(x, y, z) = \epsilon'(x, y) + \Delta\epsilon(x, y, z)$, with $\epsilon'(x, y)$ defining the original waveguide —sometimes referred as the canonical structure— and $\Delta\epsilon(x, y, z)$ and arbitrary variation of refractive index of any form.

The concept of a canonical structure

Maxwell equations for the whole system (waveguide + perturbation) are:

$$\begin{cases} \nabla \times \boldsymbol{E} = -i\omega\mu_0 \boldsymbol{H}, \\ \nabla \times \boldsymbol{H} = i\omega\epsilon_0(\epsilon' + \Delta\epsilon)\boldsymbol{E}, \end{cases} \tag{2}$$

while if we consider the waveguide only (canonical structure):

$$\begin{cases} \nabla \times \boldsymbol{E}^{(\omega)} = -i\omega\mu_0 \boldsymbol{H}^{(\omega)}, \\ \nabla \times \boldsymbol{H}^{(\omega)} = i\omega\epsilon_0\epsilon' \boldsymbol{E}^{(\omega)}, \end{cases} \tag{3}$$

with $\boldsymbol{E}^{(\omega)}$ and $\boldsymbol{H}^{(\omega)}$ the corresponding electromagnetic field solutions. These are in general different from the solutions of the general problem $\boldsymbol{E}$ and $\boldsymbol{H}$ appearing in (2). The main idea is to expand the general solution $\boldsymbol{E}$ and $\boldsymbol{H}$ into a combination of modes of the canonical structure, which forms a complete eigenbasis.

Outline

1. General Formulation
2. Data-driven expansion and dimensionality reduction
3. The CMT equations
4. General properties of CMT equations
5. General considerations
6. Reference texts

Data-driven modal expansion

The core of such a data-driven approach, is avoid proceeding to expand these solutions directly for Maxwell equations, but conversely use the following relationship, which is a special application of the reciprocity theorem of Maxwell equations introduced in the previous chapters:

Reciprocity condition

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}^{(\omega)*} + \boldsymbol{E}^{(\omega)*} \times \boldsymbol{H}) + i\omega\epsilon_0 \Delta\epsilon \boldsymbol{E}^{(\omega)*} \cdot \boldsymbol{E} = 0$$

This result is demonstrated following the same steps used in Eq. (3) of lesson 9. We now integrate this condition $\iint_{-\infty}^{\infty} dxdy$ all over the transverse space, and obtain:

$$\int dxdy \partial_z \left[\boldsymbol{E}_t \times \boldsymbol{H}_t^{(\omega)*} + \boldsymbol{E}_t^{(\omega)*} \times \boldsymbol{H}_t \right] \cdot \hat{z} + i\omega\epsilon_0 \int dxdy \boldsymbol{E}^{(\omega)*} \Delta\epsilon \boldsymbol{E} = 0 \quad (4)$$

Data-driven modal expansion

We then expand the transverse electromagnetic fields $\boldsymbol{E}_t$ and $\boldsymbol{H}_t$ as a superposition of a set of modes $\boldsymbol{\mathcal{E}}_\nu$ and $\boldsymbol{\mathcal{H}}_\nu$ of the canonical structure with propagation constant β_ν:

$$\begin{bmatrix} \boldsymbol{E}_t(x,y,z) \\ \boldsymbol{H}_t(x,y,z) \end{bmatrix} = \sum_\nu a_\nu(z) \begin{bmatrix} \boldsymbol{\mathcal{E}}_\nu(x,y) \\ \boldsymbol{\mathcal{H}}_\nu(x,y) \end{bmatrix} e^{-i\beta_\nu z} \tag{5}$$

The expansion (5) is intended complete and over both discrete $\nu = 0, 1, 2...$ guided modes and continuous (radiation) modes. To keep the notation as simple as possible, we indicate the complete expansion with a single $\sum_\nu$. Equation (5), by using the modes of the canonical structure, typically allows to express the solution of the whole structure with few components, allowing for an efficient dimensionality reduction of the problem.

Data-driven modal expansion

The next step is to consider a single mode μ of the canonical structure:

$$\begin{cases} \boldsymbol{E}_t^{(\omega)}(x,y,z) = \boldsymbol{\mathcal{E}}_\mu(x,y)e^{-i\beta_\mu z} \\ \boldsymbol{H}_t^{(\omega)}(x,y,z) = \boldsymbol{\mathcal{H}}_\mu(x,y)e^{-i\beta_\mu z} \end{cases} \tag{6}$$

and substitute Eqs. (5)-(6) into Eqs. (4). The first term on the LHS of (4) reads:

$$\int dxdy\partial_z \left[\boldsymbol{E}_t \times \boldsymbol{H}_t^{(\omega)*} + \boldsymbol{E}_t^{(\omega)*} \times \boldsymbol{H}_t\right] \cdot \hat{z} = \sum_\nu \left[\partial_z a_\nu - i(\beta_\nu - \beta_\mu)\right]$$
$$\times \iint dxdy \left[\boldsymbol{\mathcal{E}}_\nu \times \boldsymbol{\mathcal{H}}_\mu^* + \boldsymbol{\mathcal{E}}_\mu^* \times \boldsymbol{\mathcal{H}}_\nu\right] \cdot \hat{z} = \pm 4\partial_z a_\mu, \tag{7}$$

and obtained exploiting the orthogonality conditions of guided modes (see lesson 9):

$$\iint dxdy \left[\boldsymbol{\mathcal{E}}_\nu \times \boldsymbol{\mathcal{H}}_\mu^* + \boldsymbol{\mathcal{E}}_\mu^* \times \boldsymbol{\mathcal{H}}_\nu\right] \cdot \hat{z} = \pm 4\delta_{\mu\nu} \quad sign(\beta_\mu) = \pm 1 \tag{8}$$

Data-driven modal expansion

From Maxwell equations: $\nabla_\perp \times \boldsymbol{H}_t = i\omega\epsilon_0(\epsilon' + \Delta\epsilon)\boldsymbol{E}_z$. By solving for $\boldsymbol{E}_z$, we have:

$$\boldsymbol{E} = \boldsymbol{E}_t + \frac{\nabla_\perp \times \boldsymbol{H}_t}{i\omega\epsilon_0(\epsilon' + \Delta\epsilon)} = \sum_\nu a_\nu \cdot \left[\boldsymbol{\mathcal{E}}_\nu + \frac{\nabla_\perp \times \boldsymbol{\mathcal{H}}_\nu}{i\omega\epsilon_0(\epsilon' + \Delta\epsilon)}\right] e^{-i\beta_\nu z}. \quad (9)$$

From Maxwell equations of the canonical structure written for the ν-th eigenmode: $\nabla_\perp \times \boldsymbol{\mathcal{H}}_\nu = i\omega\epsilon_0\epsilon' \boldsymbol{\mathcal{E}}_{z\nu}$, we can express Eq. 10 as follows:

$$\boldsymbol{E} = \sum_\nu a_\nu \cdot \left[\boldsymbol{\mathcal{E}}_\nu + \frac{\epsilon'}{\epsilon' + \Delta\epsilon}\boldsymbol{\mathcal{E}}_{z\nu}\right] e^{-i\beta_\nu z}. \quad (10)$$

With this expression, the RHS of Eq. 4 reads:

$$\boldsymbol{E}^{(\omega)*}\Delta\epsilon\boldsymbol{E} = \sum_\nu a_\nu e^{-i(\beta_\nu - \beta_\mu)z}\Delta\epsilon\left(\boldsymbol{\mathcal{E}}^*_\mu\boldsymbol{\mathcal{E}}_\nu + \frac{\epsilon'}{\epsilon' + \Delta\epsilon}\boldsymbol{\mathcal{E}}_{z\nu}\boldsymbol{\mathcal{E}}^*_{z\mu}\right) \quad (11)$$

Outline

1. General Formulation
2. Data-driven expansion and dimensionality reduction
3. The CMT equations
4. General properties of CMT equations
5. General considerations
6. Reference texts

Final CMT equations

By combining (7) and (11), we obtain the final set of CMT equations:

CMT Equations

$$\pm\,\partial_z a_\mu(z) = -i\sum_\nu C_{\mu\nu}(z)a_\nu(z)e^{-i(\beta_\nu-\beta_\mu)z} \quad (12)$$

with coupling coefficients $C_{\mu\nu}$:

$$C_{\mu\nu}(z) = \frac{\omega\epsilon_0}{4}\iint dxdy\,\Delta\epsilon\left(\boldsymbol{\mathcal{E}}_\mu^*\boldsymbol{\mathcal{E}}_\nu + \frac{\epsilon'}{\epsilon'+\Delta\epsilon}\mathcal{E}_{z\mu}^*\mathcal{E}_{z\nu}\right), \qquad (13)$$

which depends on the spatial overlap between the guided modes profile $\boldsymbol{\mathcal{E}}_\nu$, $\mathcal{E}_{z\nu}$ of the canonical structure and the perturbation term $\Delta\epsilon(x,y,z)$. Equations (12) are exact and are derived without any approximation. The $\pm$ sign in front of the LHS terms depends on the sign of the μ-th mode propagation constant. For positive $\beta_\mu > 0$, the mode is forward propagating and the CMT equations have the $+$ sign, for backward propagating mode with $\beta_\mu < 0$, the CMT equations have the $-$ sign.

Outline

1 General Formulation

2 Data-driven expansion and dimensionality reduction

3 The CMT equations

4 General properties of CMT equations

5 General considerations

6 Reference texts

The CMT equations

1. The coupling matrix is Hermitian: $C_{\mu\nu}(z) = C^*_{\nu\mu}(z)$
2. The quantity $\sum_\mu |a_\mu|^2$ is conserved along propagation z and is an integral of motion

The result 1 is demonstrated via visual inspection of Eq. (13). The demonstration of point 2 is performed as follows. The CMT equations can be rewritten in the following vectorial form:

$$\partial_z \boldsymbol{a} = -i\underline{\underline{C}} \cdot \boldsymbol{a}, \tag{14}$$

with $\boldsymbol{a}$ a row vector containing the modal amplitudes $a_\mu(z)$ and $\underline{\underline{C}}$ a coupling matrix with elements $C_{\mu\nu}(z) \exp^{-i(\beta_\nu - \beta_\mu)z}$. We then have:

$$\partial_z(\boldsymbol{a}^\dagger \cdot \boldsymbol{a}) = \partial_z(\boldsymbol{a}^\dagger) \cdot \boldsymbol{a} + \boldsymbol{a}^\dagger \cdot \partial_z \boldsymbol{a} = i\boldsymbol{a}^\dagger \cdot \underline{\underline{C}}^\dagger \cdot \boldsymbol{a} - i\boldsymbol{a}^\dagger \underline{\underline{C}} \cdot \boldsymbol{a} = 0, \tag{15}$$

where we have used the Hermitian condition demonstrated at point 1: $\underline{\underline{C}} = \underline{\underline{C}}^\dagger$.

The CMT equations

Advanced question: what is the physical meaning of the integral of motion found: $\sum_\mu |a_\mu(z)|^2 = \text{const}$?

The CMT equations

Advanced question: what is the physical meaning of the integral of motion found: $\sum_\mu |a_\mu(z)|^2 = \text{const}$?

It represents the optical power carried out the by the guided modes, and the integral of motion represents the conservation of optical power along propagation.

Demonstration. From lesson 10, Equation (6) applied to the modal expansion (5), we have:

$$P = \frac{1}{4} \iint dxdy(\boldsymbol{E}^* \times \boldsymbol{H} + \boldsymbol{E} \times \boldsymbol{H}^*) = \sum_\mu |a_\mu(z)|^2. \tag{16}$$

This results implies that light propagating inside waveguide structures exchanges power among the modes locally at every point z along the propagation axis, maintaining the total power available $\sum_\mu |a_\mu(z)|^2$ conserved along z.

Outline

1. General Formulation
2. Data-driven expansion and dimensionality reduction
3. The CMT equations
4. General properties of CMT equations
5. General considerations
6. Reference texts

General considerations on power exchange

$$\pm\partial_z a_\mu(z) = -i\sum_\nu C_{\mu\nu} a_\nu(z) e^{-i(\beta_\nu-\beta_\mu)z} \tag{17}$$

The CMT equations decompose light dynamics as an ensemble of interacting modes that exchange power along propagation z. In order for this exchange process to occur efficiently, we need to simultaneously satisfy two conditions:

1. The off-diagonal elements of the coupling matrix should be non-zero: $C_{\mu\nu} \neq 0$ for $\nu \neq \mu$. If the coupling matrix is diagonal, then no exchange can occur among modes μ and ν. The stronger the overlap term $C_{\mu\nu}$, the stronger the amount of power exchanged at every z.
2. Phase-matching $\Delta\beta_{\mu\nu} = \beta_\nu - \beta_\mu \approx 0$. When phase matching is fulfilled, energy is transferred in phase at every z and the process if very efficient. If phase-matching does not occur, the term $e^{-i(\beta_\nu-\beta_\mu)z}$ oscillates periodically between positive and negative values, leading to zero power exchanged on average along z, regardless the amplitude of $C_{\mu\nu}$.

Outline

1. General Formulation
2. Data-driven expansion and dimensionality reduction
3. The CMT equations
4. General properties of CMT equations
5. General considerations
6. Reference texts

Reference texts

- T. Tamir, *Guided-wave optoelectronics* (Springer, 1988). Sec. 2.6
- H. Nishihara, *Optical Integrated Circuits* (McGraw Hill, 1989). Chapter 3