ECE325 Advanced Photonics

Optical resonators and for nanoscale light control
lesson 16

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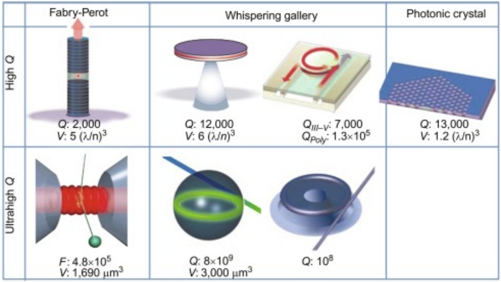
Outline

Optical resonators



Optical resonator a system possessing one or multiple resonances. A resonance is a characteristic frequency ω_0 able to accumulate electromagnetic energy in both space and time.

Resonators are typically implemented with finite volume structures that supports the generation of different types of standing waves. To evaluate the ability of a resonator to trap energy, we use the quality Q-factor.

Q-factor

The quality factor Q is an adimensional parameter defined as follows:

$$Q = \omega_0 \frac{W}{P_d}, \tag{1}$$

with:

- ω_0 resonance frequency
- W energy stored inside the resonator
- P_d power dissipated by the resonator

The evolution of the energy of the resonator is computed from the energy balance equation, which is obtained by considering that the variation of energy in time should be equal to the power dissipated by the resonator:

$$\frac{dW}{dt} = -P_d = -\omega_0 \frac{W}{Q}, \quad W(t) = W(0)e^{-\frac{\omega_0}{Q}t} \tag{2}$$

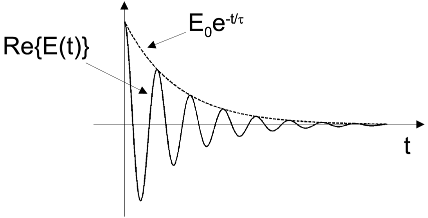
A large Q implies that the energy will be released on longer times and will strongly accumulate inside the resonator.

Q-factor

The evolution of the electric field in a resonator is $\boldsymbol{E} = \boldsymbol{E}_0 e^{i\omega_0 t - \frac{t}{\tau}}$, with τ the decaying constant. As the electromagnetic energy $W \propto |\boldsymbol{E}|^2$, we have:

$$\boldsymbol{E}(t) = \boldsymbol{E}_0 e^{i\omega_0 t - \frac{\omega_0}{2Q} t} \qquad (3)$$

with $\tau = \frac{2Q}{\omega_0}$.

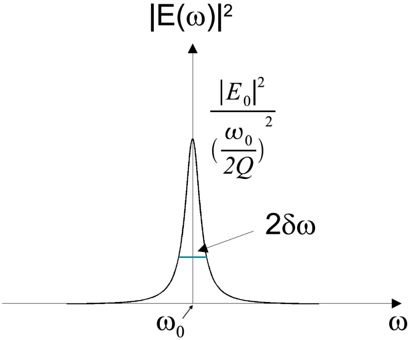


The power density spectrum of the electromagnetic field:

$$|\boldsymbol{E}(\omega)|^2 = |F[\boldsymbol{E}(t)]|^2 = \frac{1}{(\omega_0 - \omega)^2 + \left(\frac{\omega_0}{2Q}\right)^2} \qquad (4)$$

with F indicating the Fourier transform, acquires the characteristic shape of a Lorenzian function.

Q-factor



The frequency profile of the function defines all the quantities of interest for a resonator: the amplitude maximum occurs at the resonance ω_0, and the Full Width Half Maxmimum $FWHM = 2\delta\omega = \frac{\omega_0}{Q}$ defines the Quality factor. The higher the Q, the narrower the linewidth.

Outline

TDCMT

The time-domain coupled mode theory is a powerful framework to study, with relative simple equations, the dynamics of energy in open resonator coupled with the environment via multiple channels or ports, each represented with incoming (+) and reflected (-) waves. These could be, for example, modes of waveguides reaching the resonator or scattering waves impinging from an open space.

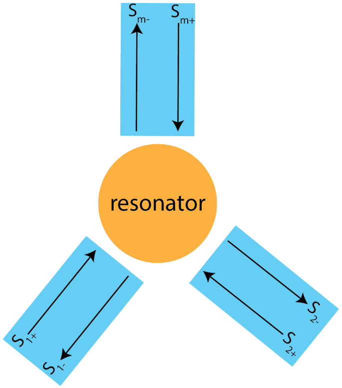


Figure: 1. TDCMT setup

TDCMT

Historically, the TDCMT equations were derived intuitively and, as such, it was long believed that this model was approximate, missing an exact link that could show the relationships between the quantities of interest in the TDCMT, and in particular the various mode amplitudes, and the spatio-temporal profile of the resonator modes. This opened the question of whether it was possible to derive a rigorous demonstration of the TDCMT, providing the missing link and addressing long standing questions about definition of resonator modes.

The answer to this question was demonstrated recently with the use of singular theory, showing a fomrulation in which the TDCMT equations are exact, and fully equivalent to Maxwell equations. We here describe the derivation of the time domain equations and their applications, reminding the reader to the References for the exact derivation from Maxwell equations.

TDCMT

The starting point of the TDCMT is to model the dynamics of the time energy inside the resonator with a generalized set of equations, and then study the relationships among the various parameters by using physical conservation laws. With reference to Fig. 1, we assume that the resonator is described by a row amplitude vector $\boldsymbol{a} = [a_1, ..., a_n]$, with $a_m(t)$ representing the amplitude of the m-th resonance. We assume that the amplitudes are normalized in such a way that the electromagnetic energy of each resonance is $|a_m(t)|^2$ and the total electromagnetic energy $W = \boldsymbol{a}^\dagger \cdot \boldsymbol{a}$.
If the resonator is linear, the evolution of the resonance amplitudes follows from a linear dynamics, which in the general case is of the form:

$$\frac{d\boldsymbol{a}}{dt} = \underline{\underline{H}} \cdot \boldsymbol{a} + \underline{\underline{K}} \cdot \boldsymbol{s}_+ \tag{5}$$

with $\underline{\underline{K}}$ couplings with the incoming waves $\boldsymbol{s}_+ = [s_{1+}, ..., s_{1m}]$, and $\underline{\underline{H}}$ a matrix characterizing the properties of resonator modes.

TDCMT

To understand the physical nature of $\underline{\underline{H}}$, we can decompose it into an skew-Hermitian matrix $i\underline{\underline{\Omega}}$, with $\underline{\underline{\Omega}}^{\dagger} = \underline{\underline{\Omega}}$, and an Hermitian part $\underline{\underline{\Omega}}^{\dagger} = \underline{\underline{\Omega}}$, obtaining $\underline{\underline{H}} = i\underline{\underline{\Omega}} - \underline{\underline{\Gamma}}$. The matrix $\underline{\underline{\Omega}}$ contains the resonant frequencies of the resonator modes, and $\underline{\underline{\Gamma}}$ the losses.
The vector $\boldsymbol{s}_{-} = [s_{1-}(t), s_{2-}(t), ..., s_{m-}(t)]$ contains the amplitudes of the time-domain component scattered from the resonator, and is expressed by using linear superposition:

$$\boldsymbol{s}_{-} = \underline{\underline{C}} \cdot \boldsymbol{s}_{+} + \underline{\underline{D}} \cdot \boldsymbol{a}, \tag{6}$$

with $\underline{\underline{C}}$ the matrix response obtained in the absence of the resonator modes ($\boldsymbol{a} = 0$), also known as scattering matrix, and $\underline{\underline{D}}$ the one obtained in the absence of impinging sources ($\boldsymbol{s}_{+} = 0$). We assume that the amplitudes of impinging and outgoing waves $s_{m\pm}$ are normalized such that $|s_{m\pm}|^2$ represents the power carried by the wave, coherently to the same normalization employed in the CMT for waveguide modes.

TDCMT

The matrices $\underline{\underline{H}}$, $\underline{\underline{K}}$, $\underline{\underline{C}}$, $\underline{\underline{D}}$ are not independent, as the dynamical system of the resonator interacting with the environment has to satisfy energy conservation. The energy balance equation reads as follows:

$$\frac{dW}{dt} = \boldsymbol{s}_+^\dagger \cdot \boldsymbol{s}_+ - \boldsymbol{s}_-^\dagger \cdot \boldsymbol{s}_-, \tag{7}$$

and implies that the time variation of total energy of the resonator equals the difference between the power injected by the system by the incoming waves and the power removed from the resonator by outgoing contributions. By substituting (5)-(6) into (7), we obtain:

$$\boldsymbol{a}^\dagger \left(-2\underline{\underline{\Gamma}} + \underline{\underline{D}}^\dagger \underline{\underline{D}}\right) \cdot \boldsymbol{a} + \boldsymbol{s}_+ \left(\underline{\underline{C}}^\dagger \underline{\underline{C}} - 1\right) \boldsymbol{s}_+ + \left[\boldsymbol{s}_+ \left(\underline{\underline{K}}^\dagger + \underline{\underline{C}}^\dagger \underline{\underline{D}}\right) \boldsymbol{a} + H.c.\right] = 0 \tag{8}$$

with $H.c.$ the Hermitian conjugate.

TDCMT

Equation (8) implies the following conditions:

$\underline{\underline{C}}^\dagger \underline{\underline{C}} = \underline{\underline{C\,C}}^\dagger = 1$ the scattering matrix is unitary

$\underline{\underline{D}} = -\underline{\underline{C}} \cdot \underline{\underline{K}}^\dagger$ defines the outgoing couplings in terms of the couplings $\underline{\underline{K}}$

$\underline{\underline{\Gamma}} = \underline{\underline{K}}^\dagger \cdot \underline{\underline{K}}/2$ defines the losses in terms of the couplings $\underline{\underline{K}}$

The complete dynamical system reads:

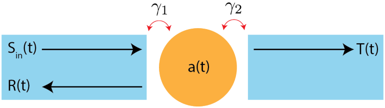
$$\begin{cases} \dot{\boldsymbol{a}} = \left(i\underline{\underline{\Omega}} - \frac{\underline{\underline{K}}\cdot\underline{\underline{K}}^\dagger}{2} \right) + \underline{\underline{K}} \cdot \boldsymbol{s}_+, \\ \boldsymbol{s}_- = \underline{\underline{C}} \cdot \left(\boldsymbol{s}_+ - \underline{\underline{K}}^\dagger \right) \cdot \boldsymbol{a}, \end{cases} \tag{9}$$

with $\dot{\boldsymbol{a}} = d\boldsymbol{a}/dt$. The system is defined once the matrices $\underline{\underline{\Omega}}$, $\underline{\underline{K}}$, $\underline{\underline{C}}$ are set.

Outline

2 ports system with one resonance

Let us consider an example of application furnished by the general system illustrated below: a 2-ports system interacting with a single resonance.



For this system, we have the following matrices:

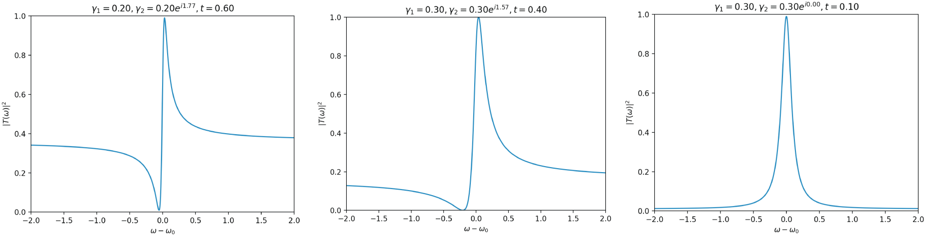
$\underline{\underline{C}} = \begin{bmatrix} r & t \\ -t^* & r^* \end{bmatrix}$, with $|r|^2 + |t|^2 = 1$, and r and t the scattering reflection and transmission, respectively, of the input signal s_{in} measured when the resonator is absent.

$\underline{\underline{K}} = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix}$, with γ_1 and γ_2 coupling coefficients of the left and right channel into the resonator.

$\underline{\underline{\Omega}} = \omega_0$, with ω_0 the resonance

2 port system with one resonance

The reflection R and transmission T output obtained by solving the TDCMT equation for this system yields a generalized Fano profile, which is capable of creating very complex responses, from perfect transmission, to perfect reflection, or both by simply acting on the coupling and scattering:



This platform provides an effective system to control light properties and engineer desired responses. This is currently a hot topic in research in the field of Mie-tronic, Methaphotonics, Inverse design, Metasurfaces, Flat-optics, Radiationless states, and other integrated components that can mold the flow of light at the nanoscale.

Outline

Reference texts

- Initial formulation: H. A. Haus, *Waves and Fields in Optoelectronics* (Prentice-Hall, 1983), Chap. 7.
- Exact formalism: Generalized Maxwell projections for multi-mode network Photonics. Sci. Rep. 10, 9038 (2020).